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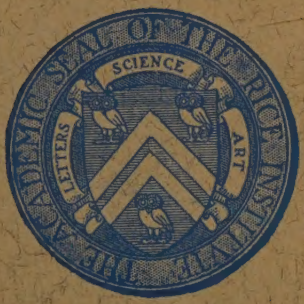

MONOGRAPH IN MATHEMATICS

COMPOSITION THEOREMS

S. MANDELBROJT

Volume XLV October, 1958 Number 3

THE RICE INSTITUTE PAMPHLET *is published at Houston, Texas, and is issued in January, April, July, and October. It was entered as second-class matter, April 15, 1915, at the post office at Houston, Texas, under the act of August 24, 1912.*

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RICE INSTITUTE PAMPHLET

Vol. XLV October, 1958 No. 3

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Published by

THE RICE INSTITUTE

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COMPOSITION THEOREMS

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Chapter I

GENERAL THEOREMS AND DEFINITIONS

1. The notion of adherent series arises when a function holomorphic in a strip containing a horizontal strip is represented by a Dirichlet series in a certain asymptotic manner. An "adherence hypothesis" is a relation between the "width" of the strip, the exponents of the Dirichlet series, and the "precision" of the representation of the function by the series. If the series $\sum d_n e^{-\lambda_n s}$ "adheres" to the function $F(s)$ in a strip Δ it is possible to majorize the moduli of the coefficients ($|d_n|$) by quantities depending only on the series of exponents ($\{\lambda_n\}$), on the maximum of $|F(s)|$ in a circle situated in Δ and of sufficiently large radius, and on the real part of the affix of the center of the circle.

In other words the concept of adherence furnishes a very wide extension of the Cauchy evaluation of the coefficients of a Taylor series. Yet this extension is of a completely different type from that furnished by the expression of Dirichlet coefficients by the integral on vertical lines. One of the most important facts in the theory of adherent series is that it is not necessary to suppose *a priori* the convergence of the Dirichlet series, and in fact, in most cases of interest the series is not *supposed* to converge anywhere. That is to say, *a priori*, no information is furnished about the convergence of the series, introduced by some functional analysis.

We shall recall the essential theorem which furnishes the principle of adherence. Some definitions are needed first.

Let $\{\lambda_n\}$ $(n \geqq 1)$ be an increasing sequence of positive numbers. For $\lambda > 0$, let $N(\lambda)$ be the number of λ_n smaller than λ, and let

$$D(\lambda) = \frac{N(\lambda)}{\lambda},$$

$$D^{\cdot}(\lambda) = \operatorname*{Sup}_{x \geqq \lambda} D(x),$$

$$\overline{D}(\lambda) = \frac{1}{\lambda} \int_0^{\lambda} D(x)\,dx,$$

$$\overline{D}^{\cdot}(\lambda) = \operatorname*{Sup}_{x \geqq \lambda} \overline{D}(x).$$

These functions are called the *density function*, the *upper density function*, the *mean density function*, and the *upper mean density function*, respectively, of *the sequence* $\{\lambda_n\}$. The quantities

$$D^* = \lim_{\lambda\to\infty} D^*(\lambda) = \limsup_{n\to\infty} \frac{n}{\lambda_n},$$

and

$$\overline{D}^* = \lim_{\lambda\to\infty} \overline{D}^*(\lambda)$$

are called the *upper density* and the *upper mean density*, respectively, of the sequence $\{\lambda_n\}$. We will assume in this Pamphlet that $D^* < \infty$. The following inequality holds: $e^{-1}D^* \leqq \overline{D}^* \leqq D^*$, [1]. Examples can be constructed where $D^* = e\overline{D}^*$ (see [1]). The function

$$\nu(D) = \operatorname*{Sup}_{\lambda\geqq 0} \lambda[\overline{D}(\lambda) - D]$$

is called the excess function of $\{\lambda_n\}$. This function is non-negative and non-increasing; we have $\nu(D) = \infty$ for $D < \overline{D}^*$ and $\nu(D)$ is finite for $D > \overline{D}^*$.

The quantities

$$\Lambda_n^* = \prod_{m\neq n} \frac{\lambda_m^2}{|\lambda_m^2 - \lambda_n^2|}$$

are finite. The sequence $\{\Lambda_n^*\}$ is called the *sequence associated with* $\{\lambda_n\}$.

Let Δ be a domain in the $s = \sigma + it$ plane such that the intersection of Δ with any half-plane $\sigma > \sigma_0$ is not empty. Let $\{d_n\}$ be a sequence of complex numbers and let $\{\lambda_n\}$ be an increasing sequence of positive numbers. Let $F(s)$ be a holomorphic and uniform function in Δ.

k being a positive integer, let $p_k(x)$ be a non-decreasing function tending to $+\infty$; $p_k(x)$ may be equal to $+\infty$ for x sufficiently large. If

$$(1) \qquad \operatorname*{Inf}_{m\geqq k} \operatorname*{Sup}_{\sigma > x, s\in\Delta} \left| F(s) = \sum_{n=1}^{m} d_n e^{-\lambda_n s} \right| \leqq e^{-p_k(x)}$$

for x sufficiently large we say that the series $\sum d_n e^{-\lambda_n s}$ represents $F(s)$ in Δ, for $m \geqq k$, with the *logarithmic precision* $p_k(x)$.

If the Dirichlet series converges for $\sigma > \sigma_0$, $p_k(x)$, $k \geq 1$, can be taken equal to $+\infty$ for $x > \sigma_0$, for any Δ described above. (It is to be noticed that we do not suppose that (1) holds for every x such that $\Delta \cap P_x$, where P_x is the half-plane $\sigma > x$, is not void, but only for x sufficiently large.) The converse of this proposition is not true; if $p_k(x) = +\infty$ for $x > \sigma_0$, $k \geq 1$ we can only assert that the series $\sum d_n e^{-\lambda_n s}$ overconverges in $\Delta \cap P_{\sigma_0}$.

The domain Δ will generally be given by $\sigma > \sigma_0$, $|t| < \pi g(\sigma)$ (σ_0 may be equal to $-\infty$), where $g(\sigma)$ is a continuous function, for $\sigma > \sigma_0$. We will set $g = \liminf_{\sigma \to \infty} g(\sigma)$. We consider here only cases where $g > 0$.

We shall give here conditions relating the functions g, p and the sequence $\{\lambda_n\}$ which allow the coefficients of the Dirichlet series to be majorized when it represents a function in Δ with a logarithmic precision p. Such a condition is called *an adherence hypothesis.*

HYPOTHESIS A $[g(\sigma), p(\sigma), \{\lambda_n\}]$. *There exists a continuous non-increasing function* $h(\sigma)$, *with* $\lim_{\sigma=\infty} h(\sigma) = h$, *such that*

$$h < g, \qquad 2\nu[h(\sigma)] < p(\sigma) + M, \qquad (M < \infty, \sigma \text{ large}),$$

and

$$(2) \qquad \int^{\infty} \{p(\sigma) - 2\nu[h(\sigma)]\} \exp\left\{-\frac{1}{2}\int^{\sigma} \frac{du}{g(u) - h(u)}\right\} d\sigma = \infty .$$

Here ν denotes the excess function of $\{\lambda_n\}$, $p(\sigma)$ is a non-decreasing function $g = \liminf_{\sigma=\infty} g(\sigma)$, $g(\sigma)$ being continuous. The lower limits of the integrals are not indicated, since they should be taken sufficiently large (the fact that the value of the integral in (2) is finite or equal to $+\infty$—these are the only alternatives when the lower limits are large—is independent of the choice of these limits, again provided they are large).

HYPOTHESIS $A^{\cdot}[g(\sigma), p(\sigma), \{\lambda_n\}]$: $g > D^{\cdot}$ *and*

$$\int^{\infty} p(\sigma) \exp\left[-\frac{1}{2}\int^{\sigma} \frac{du}{g(u) - D^{\cdot}[p(u)]}\right] d\sigma = \infty .$$

HYPOTHESIS $\overline{A}^{\cdot}[g(\sigma), p(\sigma), \{\lambda_n\}]: g > \overline{D}^{\cdot}$

$$\int^{\infty} p(\sigma) \exp\left[-\frac{1}{2}\int^{\sigma} \frac{du}{g(u) - \overline{D}^{\cdot}[p(u)]}\right] d\sigma = \infty.$$

LEMMA 1. *The hypotheses $A^{\cdot}$, $\overline{A}^{\cdot}$ are particular cases of the hypothesis A.*

Indeed for $x > 1$:

$$\begin{aligned} \nu[\overline{D}^{\cdot}(x)] &= \operatorname*{Sup}_{\lambda \geqq 0} \lambda[\overline{D}(\lambda) - \overline{D}^{\cdot}(x)] \leqq \operatorname*{Sup}_{\lambda \geqq 0} \lambda\,[\overline{D}^{\cdot}(\lambda) - \overline{D}^{\cdot}(x)] \\ &= \operatorname*{Sup}_{0 < \lambda \leqq x} \lambda[\overline{D}^{\cdot}(\lambda) - \overline{D}^{\cdot}(x)] \\ &\leqq \operatorname{Max} \lambda \left\{x^{1/2}[\overline{D}^{\cdot}(0) - \overline{D}^{\cdot}(x)],\, x[\overline{D}(x^{1/2}) - \overline{D}^{\cdot}(x)]\right\}, \end{aligned}$$

$\overline{D}^{\cdot}(\lambda)$ being a non-increasing function tending to $\overline{D}^{\cdot}$ when λ tends to ∞. Therefore

$$\nu[\overline{D}^{\cdot}(x) = o(x), \qquad (x \to \infty). \tag{3}$$

We have also

$$\begin{aligned} \nu[D^{\cdot}(x)] &= \operatorname*{Sup}_{\lambda \geqq 0} \int_0^{\lambda} [D(t) - D^{\cdot}(x)]dt \\ &\leqq \operatorname*{Sup}_{\lambda \geqq 0} \int_0^{\lambda} [D^{\cdot}(t) - D^{\cdot}(x)]dt = \int_0^{x} [D^{\cdot}(t) - D^{\cdot}(x)]dt \\ &\leqq \int_0^{x^{1/2}} [D^{\cdot}(0) - D^{\cdot}(x)]dt + \int_{x^{1/2}}^{x} [D^{\cdot}(x^{1/2}) - D^{\cdot}(x)]dt \\ &\leqq x^{1/2}[D^{\cdot}(0) - D^{\cdot}(x)] + x[\overline{D}^{\cdot}(x^{1/2}) - D^{\cdot}(x)] \end{aligned}$$

and, here too, we have

$$\nu[D^{\cdot}(x)] = o(x) \qquad (x \to \infty). \tag{4}$$

The relationship (4) shows that if $A^{\cdot}$ is satisfied then A is satisfied with $h(\sigma) = D^{\cdot}[p(\sigma)]$; the relationship (3) shows that if $\overline{A}^{\cdot}$ is satisfied then A is satisfied with $h(\sigma) = \overline{D}^{\cdot}[p(\sigma)]$.

The fundamental theorem in the theory of adherent series can be stated in the following way:

THEOREM 1. *Suppose that the following conditions are satisfied:* 1° *$\{\lambda_n\}$ is an increasing sequence of positive numbers with*

finite upper density D^*; Δ *is a domain in the* $s=\sigma+it$ *plane given by* $\sigma>a$, $|t|<\pi g(\sigma)$, *where* $g(\sigma)$ *is a continuous function of bounded variation such that* $g(\sigma)>\overline{D}^*$, *where* $\overline{D}^*$ *is the mean upper density of the sequence* $\{\lambda_n\}$;

2° $F(s)$ *is a holomorphic function in* Δ *which can be continued analytically from* Δ *to the circle* $C(s_0, \pi R)$ (*i.e.*, $|s-s_0|<\pi R$) *through a channel of diameter greater than* $2\pi\overline{D}^*$.[1]

3° *For a given positive integer* K, $\sum d_n e^{-\lambda_n s}$ *represents, with* $m>K$, $F(s)$ *in* Δ *with the logarithmic precision* $p_K(\sigma)$.

4° *The hypothesis* $A\,[g(\sigma), p_K(\sigma), \{\lambda_n\}]$ *is satisfied.*

Then

$$|d_K| \leqq (1/2)\pi^2\lambda_K R\cdot e^{2\nu(R)}\Lambda_K^* e^{\Re(s_0)} M(s_0, \pi R)$$

where

$$M(s_0, \pi R) = \operatorname*{Sup}_{s\in(C s_0, \pi R)} |F(s)|.$$

COROLLARY. *In the statement of theorem 1, the hypothesis A (in 4°) can be replaced by one of the hypotheses* A^*, $\overline{A}^*$. This follows immediately from lemma 1.

The proof of Theorem 1 will not be given here. The reader can find it in [1], where applications other than those made below and related to many branches of analysis may also be found.

2. Let E denote a closed set on the real axis R of the complex $z=x+iy$ plane. If $E=R$, let Δ_E denote the upper half-plane: $y>0$. If $E\neq R$, let Δ_E denote the complement of E with respect to the entire z-plane. Let $M(r)$ be a positive function defined for $r>0$ and let $p\geqq 1$. We say that $M(r)$ *is a function associated* (p) *with the set* E, if the class $W_p(E, M)$ of functions $\Phi(z)$ holomorphic and uniform in Δ_E, with $\Phi\not\equiv 0$, and satisfying in Δ_E the inequality[2]

$$|\Phi(z)| \leqq M(|z|)\,|y|^{1/p-1}$$

is void. The concept of "$M(r)$ associated with E" was already introduced in [2]. We shall now generalize this concept in the following fashion.

Let $\Lambda\equiv\{\lambda_n\}$ be a sequence of positive increasing integers, and let $\mathfrak{M}\equiv\{M_n\}$ be a sequence of positive numbers. We say

that *the sequence* $\mathfrak{M}$ *is associated* (p) *with the set* E *and the sequence* Λ if the following conditions are satisfied:

The class $W_p(E, \mathfrak{M}, \Lambda)$ of functions $\Phi(z)$, holomorphic and uniform in Δ_E, with $\Phi(z) \not\equiv 0$, and such that to each Φ corresponds a sequence of complex numbers $\{d_k\}$ so that the following inequalities are satisfied in Δ_E:

$$\left| \Phi(z) = \sum_{n=1}^{m} \frac{d_n}{z^{\lambda_n+1}} \right| \leqq A M_{q+1} |z|^{-q-1} |y|^{1/p-1}$$

$$(\lambda_m \leqq q < \lambda_{m+1},\ m \leqq 1),\ |\Phi(z)| \leqq A |y|^{1/p-1}, \tag{5}$$

(where A is a constant, and where q takes, for each $m \geqq 1$, all the integral values of the semi-closed interval $[\lambda_m, \lambda_{m+1})$), is void.

If a is a positive quantity we call the set E_a, defined by

$$E_a = \bigcup_{x \in E} [x - a, x + a]$$

the a-covering of E.

LEMMA 2. *If a function* $\Phi(z)$, *holomorphic and uniform in* Δ_E, *satisfies the inequalities* (5) *then for* $z \in \Delta_E$, $|y| \leqq 1$, $x \notin E_a$, $a < 1$, *the following inequalities are satisfied*:

$$\left| \Phi(z) = \sum_{K=1}^{m} \frac{d_K}{z^{\lambda_K+1}} \right| \leqq \frac{5 A M_{q+1}}{a |z|^{q+1}},\ (\lambda_m \leqq q < \lambda_{m+1}, m \geqq 1),$$

$$|\Phi(z)| \leqq \frac{5A}{a}. \tag{6}$$

On setting

$$F_q(z) = \left[\Phi(z) - \sum_{K=1}^{m} \frac{d_K}{z^{\lambda_K+1}} \right] z^{q+1},$$

one sees that it is sufficient to prove that, if a function $F(z)$, holomorphic and uniform in the strip $-a < x < a$, satisfies there the inequality $|F(z)|\,|y| < M$, then for $|y| \leqq 1$,

$$|F(iy)| \leqq 5 M a^{-1}.$$

This follows from a device of Marcel Riesz. The function $F^*(z)$

$=F(z)(z-a)(z+a)$ is bounded on the square $y=\pm a$, $x=\pm a$ by the quantity $5Ma$; our conclusion is then immediate by the maximum principle.

If Φ is holomorphic and uniform in Δ_E and satisfies there the inequalities

$$|\Phi(z)| \leqq \frac{M_n}{|y|\,|z|^n}, \qquad (n \leqq 1),$$

then it satisfies also the inequality

$$|\Phi(z)| \leqq \frac{1}{|y|\,T(|z|)},$$

where $T(r)$ is the function introduced by Ostrowski [3], $T(r)=\mathrm{Sup}_n\, r^n/M_n$; so that if $\mathfrak{M}$ is associated (p) with E and a void sequence of (positive!) integers, the function $M(r)=(T(r))^{-1}$ is associated with E in the sense given before. Therefore, at least when $M(r)$ is of the form: $M(r)=\mathrm{Inf}\, M_n/r^n$, the notion of a sequence $\mathfrak{M}$ associated (p) with E and Λ generalizes that of the notion of a function $M(r)$ associated with E.

3. We will present now some well-known results concerning the Fourier-Carleman transform.

Suppose $f\in L$, on R, and let $\hat{f}$ be its Fourier transform

$$\hat{f}(x) = \mathfrak{F}(f) = \frac{1}{\sqrt{2\pi}}\int f(u)e^{-iux}du.^3$$

$\hat{f}$ is a bounded continuous function on R. The class of function ϕ which are Fourier transforms of functions of L is called A.

The *spectrum* of f is defined as the set

$$\sigma(f) = \overline{\{u \mid \hat{f}(u) \neq 0\}}.^4$$

If we set for $z=x+iy$:

$$F^+(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} f(u)e^{-iuz}du, \quad \text{for } y > 0$$

(7)

$$F^-(z) = -\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} f(u)e^{-iuz}du, \quad \text{for } y < 0,$$

the functions F^+, F^- are analytic continuations of each other across each interval (a, b) which belongs to the set complementary to $\sigma(f)$ with respect to R (if $\sigma(f) \neq R$). In other words, there exists a function $F(z)$ holomorphic in $\Delta_{\sigma(f)}$, equal to $F^+(z)$ for $y>0$, and (if $\sigma(f) \neq R$) to $F^-(z)$ for $y<0$. The couple (F^+, F^-) constitutes *the Fourier-Carleman transform of f*. Obviously such a couple may be defined also for functions f not necessarily belonging to L. This couple shall be denoted by $\mathfrak{FC}(f)$.

Suppose then that $f \in L^p (p \geqq 1)$ ($L^\infty \equiv B$ denotes the class of essentially bounded functions). Define F^+ and F^- by (7). These functions are again holomorphic and uniform in the upper half-plane $y>0$ and lower half-plane $y<0$, respectively. If $f \in L$ and if no point of $[a, b]$ belongs to $\sigma(f)$, then $F^+(x+iy) - F^-(x-iy)$, $(y>0)$ tends to zero with y, uniformly on $[a, b]$. In a general way, if $f \in L^p$, an interval (a, b), such that F^+ can be continued into F^- across (a, b), thus

$$\lim_{y \downarrow 0} [F^+(x + iy) - F^-(x - iy)] = 0,$$

uniformly on $[a', b']$, $(a<a'<b'<b)$, is called a *concordance interval* of f. And the set complementary to the union of concordance intervals with respect to R is called *the spectrum of f*. This definition is consistent with the one given for the spectrum of a function $f \in L$.

Here again there exists a function F holomorphic and uniform in $\Delta_{\sigma(f)}$, equal to F^+ for $y>0$ and (if $\sigma(f) \neq R$) to F^- for $y<0$.

We see by Schwarz's inequality that

$$|F^+(z)| \leqq \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{0} e^{qyu} du \right)^{1/q} \left(\int_{-\infty}^{0} |f(u)|^p du \right)^{1/p},$$

$$\left(\frac{1}{p} + \frac{1}{q} = 1 \right),$$

that is

$$|F^+(z)| \leqq C(p) |y|^{-1/q} \|f\|_p,$$

(where $\|f\|_p$ is the norm of f in L_p). The same inequality holds for F^- in the half-plane $y<0$.

4. We denote by $L^p(\mathfrak{M}, \Lambda, E)$ the class of L^p functions, infinitely differentiable on R satisfying the conditions

$$f^{(n)} \in L^p, \|f^{(n)}\|_p \leqq M_n, \sigma(f) \subset E, f^{(\lambda_n)}(0) = f(0) = 0 (n \geqq 1).$$

If no condition on $\sigma(f)$ is indicated, that is to say, if $E=R$, we shall denote the class $L^p(\mathfrak{M}, \Lambda)$ [thus, f belongs to $L^p(\mathfrak{M}, \Lambda)$ if, and only if $\|f^{(n)}\|_p \leqq M_n$, $f(0)=f^{(\lambda_n)}(0)=0$]. If F is the function equal to F^+ for $y>0$, and (if $\sigma(f) \not\equiv R$) to F^- for $y<0$, we shall also write $F=(F^+, F^-)=\mathfrak{FC}(f)$. If $\sigma(f) \in E$, F is holomorphic and uniform in Δ_E.

LEMMA 3. *Let $f \in L^p(\mathfrak{M}, \Lambda)$, $(p \geqq 1)$ and let $F=\mathfrak{FC}(f)$. Let $\{\mu_n\}$ be the sequence of positive integers complementary to Λ with respect to all the positive integers.*

The following inequalities hold for $|y|>0$:

$$\left|\sqrt{2\pi}F(z) - \sum_{n=1}^{m} \frac{f^{(\mu_n)}(0)}{(iz)^{\mu_n+1}}\right|$$

$$(8) \qquad \leqq A(p)\|f^{(q+1)}\|_p |z|^{-q-1} |y|^{1/p-1}$$

$$\leqq A(p) M_{q+1} |z|^{-q-1} |y|^{1/p-1}, (\mu_m \leqq q < \mu_{m+1}, m \geqq 1);$$

$$|\sqrt{2\pi}F(z)| \leqq A(p)|y|^{1/p-1}.$$

By repeated integrations by parts of the first formula of (7) we obtain for $y>0$:

$$\sqrt{2\pi}F^+(z) = \frac{f(0)}{iz} + \frac{f'(0)}{(iz)^2} + \cdots + \frac{f^{(n-1)}(0)}{(iz)^n}$$

$$+ \frac{1}{(iz)^n}\int_{-\infty}^{0} f^{(n)}(u)e^{-izu}du.$$

Since $f(0)=f^{(\lambda_n)}(0)=0$, the above equation reduces to

$$\sqrt{2\pi}F^+(z) = \sum_{n+1}^{m} \frac{f^{(\mu_n)}(0)}{(iz)^{\mu_n+1}} + \frac{1}{(iz)^{q+1}}\int_{-\infty}^{0} f^{(q+1)}(u)e^{-iuz}du,$$

where q is any integer such that $\mu_m \leqq q < \mu_{m+1}$.

The inequality of Schwarz furnishes immediately (8) for F^+. The same inequality holds for $F^-(y<0)$, and the lemma is proved. The second inequality of (8) is immediate.

THEOREM 2. *Let* $f \in L^p(\mathfrak{M}, \Lambda, E)$. *If the sequence* $\mathfrak{M}$ *is associated* (p) *with* E *and* $\{\mu_n\}$, $\{\mu_n\}$ *being the sequence complementary to* Λ *with respect to all the positive integers, then* $f(u) \equiv 0$.

On setting $F = \mathfrak{F}\mathfrak{C}(f)$, we see, by lemma 3, and the fact that $\mathfrak{M}$ is associated (p) with E and $\{\mu_n\}$, that $F(z) \equiv 0$. From $F^+(z) \equiv 0$ it follows immediately that $f(u) = 0$ for $u < 0$, and from $F^-(z) \equiv 0$ that $f(u) = 0$ for $u > 0$.

Theorem 1 will allow us to give explicit conditions relating $\mathfrak{M}$, E and $\{\mu_n\}$ in order that $\mathfrak{M}$ be associated (p) with E and $\{\mu_n\}$. This will be done in Chapters II and III, where (Ch. III) the condition "The class $W(\mathfrak{M}, E) = W(\mathfrak{M}, E, 1)$ is void" will be related to many other conditions concerning problems apparently distinct from this one. But now, let us handle rapidly the case where $E = R$, that is to say, the case where Δ_E is the upper half-plane ($p \geqq 1$, being arbitrary). For that purpose we establish the following lemma:

LEMMA 4. *Let* $\Phi(z)$ *be holomorphic in the half-plane* $y > 0$, *and satisfy there the inequalities* (5), *and let us set* $\Psi(s) = A^{-1}\alpha^{1-(1/p)}\Phi(ie^s)$. *The series* $\sum i^{\lambda_n+1}\alpha^{1-(1/p)}A^{-1}d_n e^{-(\lambda_n+1)s}$ *represents* $\Psi(s)$ *in every* Δ^α, *defined by*

$$\sigma > \log \alpha, \quad |t| < \frac{\pi}{2}(1 - \alpha e^{-\sigma}),$$

for $m > K$, K *being any positive integer, with the logarithmic precision* $C(\sigma)$ *given by*

$$C(\sigma) = \operatorname*{Sup}_{n \geqq 1}(n\sigma - \log M_n) \tag{9}$$

and in Δ^α: $|\Psi(s)| < 1$.

It follows, indeed, from (5), $K \geqq 1$ (integer) given, that, for $y \geqq \alpha$, $m \geqq K$,

$$\left|\Phi(z) - \sum_{n=1}^{m} \frac{d_n}{z^{\lambda_n+1}}\right| \leqq A\alpha^{1/p-1} \operatorname*{Inf}_{q \geqq \lambda_k} (M_{q+1}|z|^{-q-1}).$$

It is readily seen that for r sufficiently large ($r \geqq r_K$), we have:

$$\operatorname*{Inf}_{q \geqq \lambda_k} (M_{q+1}|z|^{-q-1}) = \operatorname*{Inf}_{n \geqq 1} (M_n|z|^{-n}).$$

The transformation $z=ie^{s}$ furnishes the end of the proof.

An easy application of theorem 1 enables us to prove the following theorem.

THEOREM 3. *Let* $\{\mu_n\}$ *be the sequence complementary to* Λ *with respect to all the positive integers. Denote by* $D^{\cdot}_{\mu}(x)$, $\overline{D}^{\cdot}_{\mu}(x)$ *respectively the upper density function and the upper mean density function of* $\{\mu_n\}$. *Let* $D^{\cdot}_{\mu}$ *be the upper density of* $\{\mu_n\}$. *Let us define* $C(\sigma)$ *by* (9). *If* $D^{\cdot}_{\mu}<1/2$, *and if one of the following conditions is satisfied*

$$(10)\qquad \begin{aligned} &\int^{\infty} C(\sigma)\exp\left\{-\int^{\sigma}\frac{du}{1-2D^{\cdot}_{\mu}[C(u)]}\right\}d\sigma=\infty,\\ &\int^{\infty} C(\sigma)\exp\left\{-\int^{\sigma}\frac{du}{1-2\overline{D}^{\cdot}_{\mu}[C(u)]}\right\}d\sigma=\infty, \end{aligned}$$

then $L^{p}(\mathfrak{M},\Lambda)$ *contains only the function identically zero.*

The divergence of the first of the integrals (10) implies the divergence of the similar integral when in the expression $\exp\{-\int^{\sigma}[1-2D^{\cdot}_{\mu}[C(u)]]^{-1}du\}$ the bracket is replaced by $\{-\int^{\sigma}[1-\alpha e^{-u}-2D^{\cdot}_{\mu}[C(u)]]^{-1}du\}$. The same remark holds for the second integral. On the other hand, one sees easily that on setting, for example, $\sigma_{\alpha}=\log\alpha-\log(1-2D^{\cdot}_{\mu})+D^{\cdot}_{\mu}+1$, there exists a circle $C(\sigma_{\alpha},\pi R)$ with $D^{\cdot}_{\mu}<R$, $C(s_0,\pi R)\subset\Delta_{\alpha}$. On applying theorem 1 and lemmas 3 and 4 we have:

$$\alpha^{1-1/p}\left|f^{(n)}_{\mu}(0)\right|\leqq B(R)M^{*}_{n}e^{(\mu_n+1)\sigma_{\alpha}},\qquad (n\geqq 1),$$

where M_n^{*} is the sequence associated with the sequence $\{\mu_n\}$ (R remains constant, $B(R)<\infty$). Since $\sigma_{\alpha}\to-\infty$ as $\alpha\to 0$, $\mu_n>0$, we have $f^{(\mu_n)}(0)=0$ $(n\geqq 1)$. That $F(z)\equiv 0$ follows then from the inequalities (8) which become for $y\geqq 1$:

$$|F(z)|\leqq KT^{-1}(|z|),$$

where $T(r)=\operatorname{Sup} r^{n}/M_n$. Since each of the relations (10) implies

$$\int^{\infty}\frac{\log T(r)}{r^{2}}dr=\infty,$$

Watson's theorem (in Ostrowski's form) furnishes $F\equiv 0$ [1], and hence $f\equiv 0$.

5. We shall now prove the following lemmas.

LEMMA 5. *Let $\phi(t)$ be a non-decreasing, bounded function on $[0, \infty)$. The function*

$$L(x) = \log \int_0^\infty t^x d\phi(t)$$

is a convex function on the interval on which it is finite.

Let us set

$$\Phi(x) = \int_0^\infty t^x d\phi(t);$$

it is sufficient to prove that for $h>0$, and x such that $\Phi(x) < \infty$, $\Phi(x+2h) < \infty$, we have

$$\Phi(x + h) \leqq [\Phi(x)\Phi(x + 2h)]^{1/2}.$$

If the polynomial $P(y) = ay^2+by+c$, with real coefficients, is positive for $y>0$, then

$$a\Phi(x + 2h) + b\Phi(x + h) + c\Phi(x)$$

$$= \int_0^\infty t^x(at^{2h} + bt^h + c)d\phi(t) \geqq 0. \tag{11}$$

If we set

$$a = (1/2)\left[\frac{\Phi(x)}{\Phi(x + 2h)}\right]^{1/2}, \quad b = -1, \quad c = (1/2)\left[\frac{\Phi(x + 2h)}{\Phi(x)}\right]^{1/2},$$

$P(y)$ is non-negative and therefore, by (11),

$$[\Phi(x)\Phi(x + 2h)]^{1/2} \geqq \Phi(x + h).$$

LEMMA 6. *Let $f(x)$ be an even function, infinitely differentiable on R, such that*

$$\int |f^{(n)}|^2 dt \leqq M_n^2, \qquad (n \geqq 0),$$

$\log M_n$ being a convex function of n. Let $\log M(x)$ denote the ordinate of the polygonal line $(x \geqq 0)$ with vertices $(n, \log M_n)$. If

$$H(\omega) = \underset{A\to\infty}{\text{l.i.m.}} \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} f(t)^{-i\omega t} dt,$$

then $H(\omega)$ *is even,* $H(\omega)\omega^x$ *is integrable on* $[0, \infty)$ *for all* $x \geqq 0$, *and, for* $n \geqq 0$

$$f^{(n)}(t) = \frac{(-i)^n}{\sqrt{2\pi}} \int H(\omega)\omega^n e^{i\omega t} dt.$$

There exists moreover a constant $C < \infty$, *such that*

$$\int_0^\infty | H(\omega) | \omega^x d\omega \leqq CM(x+1), \qquad (x \geqq 0).$$

We have by the classical Plancherel's theorem

$$f(t) = \underset{A=\infty}{\text{l.i.m.}} \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} H(\omega) e^{-i\omega t} d\omega,$$

and

$$f^{(n)}(t) = (-i)^n \text{ .i.m.} \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} H(\omega)\omega^n e^{-i\omega t} d\omega. \tag{12}$$

Therefore

$$\| f^{(n)} \|_2 = \| H(\omega)\omega^n \|_2,$$

and since H is even, we have:

$$\int_0^\infty | f^{(n)}(t) |^2 dt = \int_0^\infty | H(\omega) |^2 \omega^{2n} d\omega \leqq M_n^2.$$

But, by lemma 5, $\log \int_0^\infty | H(\omega) |^2 \omega^{2x} d\omega$ is a convex function of x, therefore we may write:

$$\int_0^\infty | H(\omega) |^2 \omega^{2x} d\omega \leqq M^2(x),$$

and Schwarz's inequality gives finally, since, by convexity of $M(x)$, $\log M(x) < \log M(x+1) + O(1)$:

$$\int_0^\infty | H(\omega) | \omega^x d\omega = \int_0^1 | H(\omega) | \omega^x d\omega + \int_1^\infty | H(\omega) | \omega^x d\omega$$

$$\leqq \left[\int_0^1 |H(\omega)|^2 \omega^{2x} d\omega\right]^{1/2}$$

$$+ \left(\int_1^\infty \frac{d\omega}{\omega^2}\right)^{1/2} \left[\int_1^\infty |H(\omega)|^2 \omega^{2(x+1)} d\omega\right]^{1/2}$$

$$\leqq M(x) + M(x+1) \leqq CM(x+1).$$

In particular, $H(\omega)\omega^x \in L$ on $[0, \infty)$ for $x \geqq 0$, which allows us to write (12).

Chapter II

COMPOSITION THEOREMS

1. The present chapter is devoted to the presentation of "composition theorems." We begin by the first theorems concerning composition of infinitely differentiable functions established in [4] and [1]. The idea of "multiplying" the spectra of the composed functions, which further emphasizes the similarity of this kind of theorem to the famous theorem of Hadamard on the composition of singularities of Taylor series, will be introduced later.

Let Λ and Λ' be two complementary sequences of positive even integers ($\Lambda \cap \Lambda' =$ void, $\Lambda \cup \Lambda' =$ the set of all the positive even integers); and let $\mathfrak{M}$ and $\mathfrak{M}'$ be sequences of positive numbers. As before, we denote by $L^p(\mathfrak{M}, \Lambda)$, $L^p(\mathfrak{M}', \Lambda')$ respectively the classes $L^p(\mathfrak{M}, \Lambda, R)$, $L^p(\mathfrak{M}', \Lambda', R)$; that is to say, for instance, $L^p(\mathfrak{M}, \Lambda)$ is the class of infinitely differentiable functions on R defined by: $f \in L^p(\mathfrak{M}, \Lambda)$ if, and only if, $f^{(n)} \in L^p (n \geqq 0)$, $\|f^{(n)}\|_p \leqq M_n (\{M_n\} = \mathfrak{M})$, $f(0) = f^{(\lambda_n)}(0) = 0 (n \geqq 1) (\{\lambda_n\} = \Lambda)$; $(\mathfrak{M}' = \{M_n'\}, \Lambda' = \{\lambda_n'\})$. We shall indicate conditions, relating the two sequences $\mathfrak{M}$, $\mathfrak{M}'$, which imply that one of the two classes $L^p(\mathfrak{M}, \Lambda)$, $L^p(\mathfrak{M}', \Lambda')$ does not contain even functions not identically zero.

We shall deal only with the cases $p = 2$, $p = \infty$. *We shall suppose in this chapter that* $\log M_n$ *and* $\log M_n'$ *are convex functions of* n.

Lemma 7. *Let* ϕ *be an even function belonging to* $L^2(\mathfrak{M}, \Lambda)$, Λ *being a sequence of even positive integers. Let* $F(s)$ $(s = \sigma + it)$ *be a function holomorphic in the strip* $|t| < a\pi$, *and let* $p(t)$ *be an increasing continuous function such that*

$$\lim_{t = a\pi} p(t) = \infty,$$

and

$$\int_0^{a\pi} (a\pi - t) dp(t) < \infty.$$

Suppose that there exists a sequence of complex numbers $\{d_n\}$ *such that, for* $|t| < a\pi$

$$\left| F(s) - \sum_{n=1}^{m} d_n e^{-\lambda_n s} \right| \leqq M_q' e^{-qx+p(|t|)-bx}, \quad (\lambda_m \leqq q < \lambda_{m+1},\ m \geqq 1), \tag{13}$$

$$|F(s)| \leqq M_0' e^{p(|t|)-cx},$$

where b *and* c *are constants with* $b \geqq 0$, $0 \leqq c < \lambda_1$. *Suppose moreover*:

$$\sum \left(\frac{M_n M_n'}{M_{n+1} M_{n+1}'} \right)^{1/2a} = \infty. \tag{14}$$

Then, either $\phi \equiv 0$, *or* $F \equiv 0$; *if* $F \equiv 0$ *then also* $d_n = 0 (n \geqq 1)$.[5]

It can be assumed without loss of generality that $\|\phi\|_2 \leqq M_0$ with $M_1 \leqq (M_0 M_2)^{1/2}$. In other words, that log M_n is a convex function for $n \geqq 0$.

Suppose then $\phi \not\equiv 0$. We must then prove that $F \equiv 0$, with $d_n = 0 (n \geqq 1)$.

Let us set

$$H(\omega) = \text{l.i.m.}_{A=\infty} \int_{-A}^{A} \phi(x) e^{ix\omega} dx,$$

$$K(x) = H(e^x) e^x.$$

$H(\omega)$ is an even function, and, by lemma 6, we may write:

$$\int K(x) e^{\lambda_n x} dx = \int_0^\infty H(\omega) \omega^{\lambda_n} d\omega = (-i)^{\lambda_n} \pi \phi^{(\lambda_n)}(0) = 0, \tag{15}$$

$$(n \geqq 1),$$

and

$$\int |K(x)| e^{x\xi} dx \leqq CM(\xi + 1), \qquad (\xi \geqq 0), \tag{16}$$

where log $M(x)$ denotes the ordinate of the point of abscissa x of the polygonal line with vertices at $(n, \log M_n)$, $(n \geqq 0)$.

We shall now use a method which was first used by Fuchs [5]

in a proof generalizing a theorem of the author of this Pamphlet.

For $|t| < a\pi$, x real, we can write:

$$(17)\quad \left| F(s-x) - \sum_{n=1}^{m} d_n e^{-\lambda_n s} e^{\lambda_n x} \right| \leqq M_\sigma' e^{-q\sigma + p(|t|) + qx - b\sigma + bx} \quad (\lambda_m \leqq q < \lambda_{m+1},\ m \geqq 1).$$

Since, by (15)

$$\int K(x) \sum_{n=1}^{m} d_n e^{-\lambda_n (s-x)} dx = 0,$$

on using (16), with $\xi = q+b$, (17) furnishes the inequality

$$(18)\quad \left| \int F(s-x)K(x)dx \right| \leqq M_q' e^{-(q+b)\sigma + p(|t|)} \int |K(x)| e^{(q+b)x} dx \leqq C_1 M_{q+k} M_{q+k}' e^{-(q+b)\sigma + p(|t|)}, \qquad (q \geqq \lambda_1,\ |t| < a\pi),$$

where k is the smallest integer not smaller than $1+b$, and where C_1 is a constant (the existence of C_1 follows from the convexity also of $\log M_n'$ as a fundtion of n). The function

$$(19)\quad F_2(s) = \int F(s-x)K(x)dx$$

is holomorphic in the strip $S \equiv (|t| < a\pi)$. Indeed, D being any compact, with $D \subset S$, we have by the second inequality in (13); for $s \in D$:

$$|t| \leqq t_0 < a\pi, \qquad \sigma > \alpha,$$

$$(20)\quad \int |F(s-x)K(x)|\, dx \leqq M_0' e^{p(t_0)} e^{-c\alpha} \int |K(x)|\, e^{cx} dx < \infty;$$

(that is to say, the integrated expression in the left number is majorized in D by a positive L function, independent of s).

Let us set $\pi g(\sigma) = p^{-1}(\sigma)$, where $p^{-1}(\sigma)$ is the inverse function

of $p(t)$, $p^{-1}(p(t))=\sigma$; $g(\sigma)$ tends increasingly to a as σ tends to infinity and

$$\int^{\infty}[a-g(\sigma)]d\sigma<\infty. \tag{21}$$

If we choose σ_0 sufficiently large, and denote by Δ the domain $\sigma>\sigma_0$, $|t|<\pi g(\sigma)$, the function $\Phi(s)=e^{(b-k-1)s}F_1(s)$ is holomorphic in Δ, and, by (18), satisfies there the inequalities:

$$\begin{aligned}|\Phi(s)| &\leqq C_1M_{q+k}M'_{q+k}e^{-q\sigma+p(|t|)-(k+1)\sigma}\\ &\leqq C_1M_{q+k}M_{q+k}e^{-(q+k)\sigma}, \qquad (q\geqq\lambda_1).\end{aligned}$$

It follows then from a special, rather simple, case of a generalized version of the solution of Watson's problem (see theorem 2.4.II in [1] and [6])[6], that $\Phi(s)\equiv 0$, that is to say, $F_1\equiv 0$. It follows from (13) and (15) that for any b' with $c<b'<\lambda_1$:

$$e^{b'x}F(x)=0(e^{-l|x|}), \qquad (|x|\to\infty, l>0).$$

Therefore, $e^{b'x}F(x)\in L$. It follows from (16) that $e^{b'x}K(x)\in L$. If F^* and K^* denote respectively the Fourier transforms of $e^{b'x}F(x)$ and $e^{b'x}K(x)$, $\sqrt{2\pi}F^*\cdot K^*$ is the Fourier transform of

$$e^{b'u}\int F(u-t)K(t)dt=e^{b'u}F_1(u).$$

Since $F_1\equiv 0$, we have $F^*K^*\equiv 0$. But K^* is the restriction to R of the function $K^*(z)$ $(z=x+iy)$, given by:

$$\begin{aligned}K^*(z)&=\frac{1}{\sqrt{2\pi}}\int K(u)e^{(b'-is)u}du\\ &=\frac{1}{\sqrt{2\pi}}\int_0^{\infty}H(\omega)\omega^{b'-is}d\omega,\end{aligned}$$

this function being holomorphic, as we see by this formula, for $y>-b'$. $H(\omega)$ is distinct from zero on a set of positive measure, since ϕ is not identically zero (ϕ is continuous!). The same is therefore true for K and K^*. Since K^* is analytic on R, $K^*(u)=0$ only on a discrete set of points. It follows then from $F^*K^*\equiv 0$ that $F^*\equiv 0$. Thus $F(s)\equiv 0$. If we set then in (13) $F(s)\equiv 0$,

we get for every $m \geqq 1$:

$$\sum_{1}^{m} d_n e^{-\lambda_n \sigma} = 0(e^{-\lambda_m \sigma}), \qquad (\sigma \to \infty),$$

which proves that $d_n = 0$, $(n \geqq 1)$. Our lemma is proved.

LEMMA 8. *In lemma 7, $L^2(\mathfrak{M}, \Lambda)$ can be replaced by $L^\infty(\mathfrak{M}, \Lambda)$, provided the supplementary condition $b < \lambda_2 - \lambda_1$ is added.*

We have now:

$$|\phi^{(n)}(x)| \leqq M_n, \qquad (n \geqq 0),$$
$$\phi(0) = \phi^{(\lambda_n)}(0) = 0, \qquad (n \geqq 1).$$

Let us set

$$g(x) = \frac{1}{x} \int_0^x \phi''(t)dt.$$

$g(x)$ is an even function, and

$$g^{(n)}(x) = x^{-n-1} \int_0^x \phi^{(n+2)}(t) t^n dt.$$

This is seen by successive integrations by parts. We see also that

$$g^{(\lambda_n - 2)}(0) = 0, \qquad (n \geqq 1).$$

If we suppose, as in the proof of lemma 7, that $\mathfrak{M}$ is a non-decreasing sequence (which is no restriction), we may write:

$$\int_0^\infty |g^{(n)}(x)|^2 dx \leqq \int_0^1 \left| x^{-n-1} \int_0^x M_{n+2} t^n dt \right|^2 dx$$
$$+ \int_1^\infty x^{-2(n+1)} \left| \phi^{(n+1)}(x) x^n - n \int_0^x \phi^{(n+1)}(t) t^{n-1} dt \right|^2 dx$$
$$\leqq M_{n+2}^2 \frac{1}{(n+1)^2} + 4M_{n+1}^2 \leqq d^2 M_{n+2}^2, \qquad (n \geqq 0),$$

where d is a positive constant.

Lemma 7, where $F(s)$ is replaced by $[F(s) - d_1 e^{-\lambda_1 s}]e^{2s}$, λ_n by

$\lambda_{n+1}-2(n\geqq 1)$, M_0' by M_{λ_1}' and c by $b+\lambda_1-2$, shows that one of the functions g or F is identically zero, which proves the lemma.

Lemma 7 may be generalized in the following fashion:

LEMMA 9. *Let* ϕ_k $(k=1, 2, \cdots, N)$ *belong to the class* $L^2(\mathfrak{M}^{(k)}, \cdot\Lambda^{(k)})$, *where* $\Lambda^{(k)}$ *are sequences of even positive integers, and let* $\bigcup_{1\leq k\leq N}\Lambda^{(k)}=\Lambda$. *Let* $F(s)$ *satisfy all the conditions of lemma* 7, *and suppose that*

$$\sum\left[\frac{M_n^{(1)}M_n^{(2)}\cdots M_n^{(k)}M_n'}{M_{n+1}^{(1)}M_{n+1}^{(2)}\cdots M_{n+1}^{(k)}M_{n+1}'}\right]^{1/2\alpha}=\infty.$$

Then either one of the functions ϕ_k *is identically zero, or* $F\equiv 0$; *if* $F\equiv 0$ *then also* $d_n=0(n\geqq 1)$.

If we replace in the beginning of the proof of lemma 7, ϕ by ϕ_1, and, correspondingly, H and K by H_1 (defined by ϕ_1 as H was defined by ϕ) and K_1, $\{\lambda_n\}=\Lambda$ by $\{\lambda_n^{(1)}\}=\Lambda^{(1)}$, $\{M_n\}=\mathfrak{M}$ by $\{M_n^{(1)}\}=\mathfrak{M}^{(1)}$, we see immediately that (18) should be replaced, because of (15) (with the corresponding changes), by

$$\left|\int F(s-x)K_1(x)dx-\sum_{n=1}^{m}d_n\left[\int K_1(x)e^{\lambda_n^{(1)}x}dx\right]e^{-\lambda_n^{(1)}s}\right|$$

$$=\left|\int F(s-x)K_1(x)dx-\sum_{n=1}^{\mu_m'}d_n'\int K_1(x)e^{\mu_n x}dxe^{-\mu_n s}\right|$$

$$\leqq C_1M_{q+k}^{(1)}M_{q+k}'e^{-(q+b)\sigma+p(|s|)},\quad(\mu_p'\leqq q\leqq\mu_{p+1}'),$$

where $\mu=\{\mu_n\}$ is the sequence complementary to $\Lambda^{(1)}$ with respect to all the positive even integers, and where μ_m' is the largest μ_n smaller than $\lambda_m^{(1)}$. On setting

$$d_n\int K_1(x)e^{\lambda_n^{(1)}x}dx=d_n^{(1)},$$

$$\int F(s-x)K_1(x)dx=F_1(s),$$

$$\int K_1(x)e^{cx}dx=A,$$

$F_1(s)$ satisfies, this time, in $|t| < \pi a$, the inequalities

$$\left| F_1(s) - \sum_1^m d_n^{(1)} e^{-\mu_n s} \right|$$

$$\leqq C_1 M_{q+k}^{(1)} M'_{q+k} e^{-q\sigma - b\sigma + p(|t|)} (\mu_m \leqq q < \mu_{m+1}, m \geqq 1)$$

and

$$|F_1(s)| \leqq A M'_0 e^{p(|t|) - c\sigma}.$$

Thus we have reduced the number of functions, ϕ, from N to $N-1$. One sees then easily how to continue our proof.

Lemma 8 may, of course, also be generalized. We have then the following lemma.

LEMMA 10. *In lemma* 9, $L^2(\mathfrak{M}^{(k)}, \Lambda^{(k)})$ *may be replaced by* $L^\infty(\mathfrak{M}^{(k)}, \Lambda^{(k)})$, *provided the supplementary condition* $b < \min_{1 \leq k \leq N} (\lambda_2^{(k)} - \lambda_1^{(k)})$ *is added.*

We leave to the reader the proof of this lemma.

2. Lemmas 7 and 8 lead to the following theorem [1].

THEOREM 4. *If* $\log M_n$ *and* $\log M'_n$ *are convex functions of* n, *and*

$$\sum \frac{M_n M'_n}{M_{n+1} M'_{n+1}} = \infty, \tag{22}$$

and if Λ *and* Λ' *are two sequences of even positive integers complementary with respect to all even positive integers, then, for* $p = 2$, *and* $p = \infty$, *one of the classes* $L^p(\mathfrak{M} \cdot \Lambda)$, $L^p(\mathfrak{M}', \Lambda')$ *contains no other even function than the function identically zero.*

We recall that $\mathfrak{M} = \{M_n\}$, $\mathfrak{M}' = \{M'_n\}$; (we recall also that, at least for $p = \infty$, it is essential to suppose for a function f of the class that also $f(0) = 0$, which is, by the way, contained in the definition of $L^p(\mathfrak{M}, \Lambda)$; but it is not necessary to suppose $f(0) = 0$ when $p = 2$). Let us first suppose $p = 2$.

Let $g \in L^2(\mathfrak{M}', \Lambda')$, and let us set for $\Delta \equiv (|t| < \pi/2)$:

$$G(s) = \int_0^\infty [g(x) - g(0)] e^{-xe^{\sigma}} dx. \tag{23}$$

We have $\int_0^\infty e^{-xe^s}dx = e^{-s}$, and $\int_0^\infty g(x)e^{-xe^s}dx$ represents also a function holomorphic in Δ, which can be seen by an easy application of Schwarz's inequality (since $g \in L^2$).

After $q+1$ integrations by parts ($\lambda_n \leqq q < \lambda_{n+1}$), we get

$$G(s) = \sum_{k=1}^{q} g^{(\lambda_k)}(0)e^{-(\lambda_k+1)s} + e^{-(q+1)s}\int_0^\infty g^{(q+1)}(x)e^{-xe^s}dx,$$

since to every $k \geqq 1$ corresponds a sequence $\{x_j^{(k)}\}$ such that

$$\lim_{j=\infty} x_j^{(k)} = \infty, \qquad \lim_{j=\infty} g^{(k-1)}(x_j^{(k)})e^{-x_j^{(k)}e^s} = 0,$$

and therefore, the non-integrated terms corresponding to the kth integration by parts are $g^{(k-1)}(0)e^{-ks}$, if $k>1$, and zero if $k=1$. We have also $g^{(\lambda_n)}(0) = 0$ $(n \geqq 1)$.

From $\|g^{(n)}\|_2 \leqq M_n'$ $(n \geqq 0)$ it follows that, for $s \in \Delta$

$$\left| G(s) - \sum_{k=1}^{n} g^{(\lambda_k)}(0)e^{-(\lambda_1+1)s} \right|$$

$$\leqq e^{-(q+1)\sigma}\left| \int_0^\infty e^{-xe^s} g^{(q+1)}(x)dx \right| \leqq e^{-(q+1)\sigma}\int_0^\infty e^{-x\Re e^s} \left| g^{(q+1)}(x) \right| dx$$

$$\leqq e^{-(q+1)\sigma}\left[\int_0^\infty e^{-2x\Re e^s}dx\right]^{1/2}\left[\int_0^\infty \left| g^{(q+1)}(x) \right|^2 dx\right]^{1/2}$$

$$\leqq e^{-(q+1)\sigma}M_{q+1}'[\sqrt{2}(e^\sigma \cos t)]^{-1/2}, \qquad (\lambda_n \leqq q < \lambda_{n+1}).$$

We have therefore in Δ:

$$\left| G(s)e^s - \sum_{1}^{n} g^{(\lambda_k)}(0)e^{-\lambda_k s} \right|$$

$$< M_{q+1}'e^{-q\sigma}(e^\sigma \cos t)^{-1/2}, \quad (\lambda_n \leqq q < \lambda_{n+1}).$$

An integration of (23) by parts gives also:

$$\left| G(s)e^s \right| \leqq M_1'(e^\sigma \cos t)^{-1/2} \qquad |t| < \frac{\pi}{2}.$$

It follows from (22) that

$$\sum \frac{M_n}{M_{n+1}} \frac{M_{n+1}'}{M_{n+2}'} = \infty.$$

It follows then from lemma 7 that, if there exists a function ϕ of $L^2(\mathfrak{M}, \Lambda)$ not identically zero, then we must have $G(s) \equiv 0$, and $g(x) \equiv g(0)$; but from $g \in L^2$ follows then that $g \equiv 0$, which proves the theorem for $p = 2$. (The hypotheses of lemma 7 are satisfied with $F(s) = G(s)e^s$, $p(t) = -1/2 \log \cos t$, $b = c = a = 1/2$.) In the case $p = \infty$ we shall set

$$G(s) = \int_0^\infty g(x) e^{-xe^s} dx,$$

and we will find in Δ:

$$\left| G(s)e^s - \sum_{k=1}^{n} g^{(\lambda_k)}(0) e^{-\lambda_k s} \right|$$

$$\leqq M'_{q+1} e^{-qs} (e^\sigma \cos t)^{-1} \quad (\lambda_n \leqq q < \lambda_{n+1},\ n \geqq 1),$$

and

$$\left| G(s)e^s \right| \leqq M'_1 (e^\sigma \cos t)^{-1}.$$

We complete the proof as before, basing ourselves, this time, on lemma 8. (The hypotheses of lemma 8 are then satisfied with $F(s) = G(s)e^s$, $p(t) = -\log \cos t$, $a = b = 1/2$, $c = 1$). If instead of lemmas 7 and 8 we use lemmas 9 and 10 we may prove the following theorem.

THEOREM 5. *If* $\log M_n^{(k)}$ $(1 \leqq k \leqq N)$ *are convex functions of* n, *and if*

$$\sum_n \prod_{1 \leqq k \leqq N} \frac{M_n^{(k)}}{M_{n+1}^{(k)}} = \infty$$

then, for $p = 2$ *and* $p = \infty$, *one of the* N *classes* $L^p(\mathfrak{M}^{(k)} \cdot \Lambda^{(k)})$, $(1 \leqq k \leqq N)$, *where the sequences of even, positive integers* $\Lambda^{(k)}$ *are such that* $\bigcup_{1 \leqq k \leqq N} \Lambda^{(k)}$ *is the set of all even positive integers, contains no other even function than the function identically zero.*

3. There is a dual relationship between the existence of a non-trivial function (that is to say not identically zero) of a class $L^p(\mathfrak{M}, \Lambda)$ and a non-trivial function holomorphic in $|t| < (\pi/2)$, admitting there an asymptotic representation by a series of the form $\sum d_n e^{-\lambda'_n s}$, Λ' being complementary to Λ with

respect to the set of all positive integers, and characterized (in a way similar to that given, say, by formula (6)) by the sequence $\mathfrak{M}$.[8]

One may then ask: does a composition theorem analogous to theorem 4 hold for functions having asymptotic representations? As a matter of fact one could establish such a theorem as a consequence of theorem 4. But, a direct proof, which does not involve infinitely differentiable functions, can be given. Before we state the theorem we shall prove the following lemmas [8].

LEMMA 11. *If $F(s)$ and $\Phi(s)$ are holomorphic in $|t|<(\pi/2)$, continuous on $|t|\leqq(\pi/2)$ and defined for $t\neq(\pi/2)(2k+1)$, k being any integer, so as to have period πi, if there exist positive constants M, M' such that $|F(s)|<M$, $|\Phi(s)|<M'$ in $|t|<(\pi/2)$, and if*

$$H_c(w)=\frac{1}{\pi i}\int_{\Gamma_c(w)}F(s)\Phi(w-s)ds,$$

where the path of integration $\Gamma_c(w)$, $w=u+iv$, proceeds along straight line segments joining $c-(\pi i/2)$, $u-c-(\pi i/2)$, $u-c+(\pi i/2)$, and $c+(\pi i/2)$, then $H_c(w)$ is also given by

$$\frac{1}{\pi i}\int_{\Gamma_c(w)}F(w-s)\Phi(s)ds,$$

and is holomorphic in the strip $0<v<\pi$.

Let $0<v<\pi$; $F(s)\Phi(w-s)$ is holomorphic, as a function of s, for $(\pi/2)<t<v+(\pi/2)$, and for $v-(\pi/2)<t<(\pi/2)$; it is continuous on the boundary of each of these strips. We see by Cauchy's theorem that

$$H_c(w)=\frac{1}{\pi i}\int_{c-\pi i/2}^{w-(c+\pi i/2)}+\frac{1}{\pi i}\int_{w-(c+\pi i/2)}^{c+\pi i/2}F(s)\Phi(w-s)ds,$$

the integration being along straight line segments.

Let $\Gamma_c'(w)$ denote the path traversed by $w-s$ as s moves along $\Gamma_c(w)$ in the opposite sense. It proceeds along straight line segments joining

$$w-\left(c+\frac{\pi i}{2}\right),\quad c+i\left(v-\frac{\pi i}{2}\right),\quad c+i\left(v+\frac{\pi}{2}\right)$$

and

$$w-\left(c-\frac{\pi i}{2}\right).$$

Again, by Cauchy's theorem:

$$\int_{\Gamma'_c(w)} F(s)\Phi(w-s)ds = \int_{w-(c+\pi i/2)}^{c+\pi i/2} + \int_{c+\pi i/2}^{w-(c-\pi i/2)} F(s)\Phi(w-s)ds.$$

But

$$\int_{c+\pi i/2}^{w-(c-\pi i/2)} F(s)\Phi(w-s)ds = \int_{c-\pi i/2}^{w-(c+\pi i/2)} F(s)\Phi(w-s)ds$$

since $F(s)\Phi(w-s)$ has period πi.

Therefore

$$\int_{\Gamma_c(w)} F(s)\Phi(w-s)ds = \int_{\Gamma_{c'}(w)} F(s)\Phi(w-s)ds$$

$$= \int_{\Gamma_c(w)} F(w-s)\Phi(s)ds.$$

Let now $w_0=u_0+iv_0$, $0<v_0<\pi$; $\epsilon_1=1/2$ min $(\pi-v_0, v_0)$; and let $\{\epsilon_n\}$ be a sequence of positive numbers decreasing to zero. Let then

$$H_n^*(w) = \frac{1}{\pi i}\int_{c-\pi i/2}^{w-c-i(\pi/2+\epsilon_n)} + \frac{1}{\pi i}\int_{w-c-i(\pi/2-\epsilon_n)}^{c+\pi i/2} F(s)\Phi(w-s)ds,$$

the integration being always along straight line segments. For $w\in C(w_0, \epsilon_1)$(i.e. $|w-w_0|<\epsilon_1$) and for s on the first part of the path of integration:

$$\frac{\pi}{2}+\epsilon_n \leqq \Im(w-s) \leqq v+\frac{\pi}{2} < v_0+\frac{\pi}{2}+\epsilon_1 \leqq \frac{3\pi}{2}-\epsilon_1;$$

on the second part:

$$\frac{\pi}{2}-\epsilon_n \geqq \Im(w-s) \geqq v-\frac{\pi}{2} > v_0-\frac{\pi}{2}-\epsilon_1 > \epsilon_1-\frac{\pi}{2}.$$

Therefore H_n^* is holomorphic in $C(w_0, \epsilon_1)$. We have by Cauchy's theorem:

$$H_c(w) - H_n^*(w) = \frac{1}{\pi i}\int_{w-c-i(\pi/2+\epsilon_n)}^{w-c-i(\pi/2-\epsilon_n)} F(s)\Phi(w-s)ds.$$

Hence $|H_c(w) - H_n^*(w)| \leqq 2MM'\epsilon_n/\pi$, and H_n^* tends uniformly to $H_c(w)$ in $C(w_0, \epsilon_1)$. $H_c(w)$ is then holomorphic in this circle, and therefore in the strip $0 < v < \pi$.

LEMMA 12. *Let F and Φ satisfy the same conditions as in lemma 9. Let us suppose, moreover, that*

$$\left|F(s) - \sum_1^m d_n e^{-2ns}\right| < M_q e^{-qs}$$

$$\left|\Phi(s) - \sum_1^m d_n' e^{-2ns}\right| < M_q' e^{-qs} \qquad (2m \leqq q < 2m+1), \tag{24}$$

where $\{M_n\}$, $\{M_n'\}$ are positive sequences, $\log M_n$, $\log M_n'$ being convex functions of n, and $M_n \geqq a > 0$, $M_n' \geqq a > 0$ $(n \geqq 1)$.

Then there exists a positive constant K, such that for $u < c$, $0 < v < \pi$, $(w = u + iv)$ the inequality

$$\left|H_c(w) - \sum_1^m d_n d_n' e^{-2nw}\right| < KM_{q+1}M_{q+1}' e^{-2qu}, \qquad (2m \leqq q < 2(m+1)),$$

holds for $u < c$, $0 < v < \pi$.

We may write:

$$H_c(w) - \sum_1^m d_n d_n' e^{-2nw}$$

$$= \frac{1}{\pi i}\int_{\Gamma_c(w)} \left[F(s)\Phi(w-s) - \sum_1^m d_n e^{-2ns}\sum_1^m d_n' e^{-2n(w-s)}\right] ds$$

$$= \frac{1}{\pi i}[A + B - C],$$

where

$$A = \int_{\Gamma_c(w)} \left[F(s) - \sum_1^m d_n e^{-2ns} \right] \Phi(w - s)ds,$$

$$B = \int_{\Gamma_c(w)} \left[\Phi(w - s) - \sum_1^m d_n' e^{-2n(w-s)} \right] F(s)ds$$

$$= \int_{\Gamma(w)} \left[\Phi(s) - \sum_1^m d_n' e^{-2ns} \right] F(w - s)ds,$$

and

$$C = \int_{\Gamma_c(w)} \left[F(s) - \sum_1^m d_n e^{-2ns} \right] \left[\Phi(w - s) - \sum_1^m d_n' e^{-2n(w-s)} \right] ds.$$

On denoting by Γ_c the closed path obtained by adding to $\Gamma_c(w)$ the straight line segment $c+(\pi i/2)$, $c-(\pi i/2)$, we can write

$$A = \int_{\Gamma_c} + \int_{c-\pi i/2}^{c+\pi i/2} \left[F(s) - \sum_1^m d_n e^{-2ns} \right] \Phi(w - s)ds.$$

The integrand in A being holomorphic as a function of s in the strips $-(\pi/2) < t < v-(\pi/2)$, $v-(\pi/2) < t < (\pi/2)$ and continuous on their boundaries, $F(s)$ being holomorphic when $t = v-\pi/2$, we can write by Cauchy's theorem

$$\int_{\Gamma_c} \left[F(s) - \sum_1^m d_n e^{-2ns} \right] \Phi(w - s)ds$$

$$= \int_{u-c}^{c} \left[F\left(\sigma + i\left(v - \frac{\pi}{2}\right)\right) - \sum_1^m d_n e^{-2n(\sigma + i(v-\pi/2))} \right]$$

$$\cdot \left[\Phi\left(u - \sigma - \frac{i\pi}{2}\right) - \Phi\left(u - \sigma + \frac{\pi i}{2}\right) \right] d\sigma$$

$$= \int_{u-c}^{c} \left[F\left(u - \tau + i\left(v - \frac{i\pi}{2}\right)\right) - \sum_1^m d_n e^{-2n(u-\tau + i(v-\pi/2))} \right]$$

$$\cdot \left[\Phi\left(\tau - \frac{\pi i}{2}\right) - \Phi\left(\tau + \frac{\pi i}{2}\right) \right] d\tau.$$

But, since

$$\Phi\left(\tau-\frac{\pi i}{2}\right)-\Phi\left(\tau+\frac{\pi i}{2}\right)=\Phi\left(\tau-\frac{\pi i}{2}\right)-\sum_1^m d_n' e^{-2n(\tau-\pi i/2)}$$
$$-\left[\Phi\left(\tau+\frac{\pi i}{2}\right)-\sum_1^m d_n' e^{-2n(\tau+\pi i/2)}\right],$$

we may write by (24)

$$\left|\Phi\left(\tau-\frac{\pi i}{2}\right)-\Phi\left(\tau+\frac{\pi i}{2}\right)\right|<2M_p' e^{-p\tau} \tag{26}$$

for every $p\geqq 0$ $(M_0'=M')$.

If follows from (25) and (24) that

$$\left|\int_{\Gamma_c}\left[F(s)-\sum_1^m d_n e^{-2ns}\right]\Phi(w-s)ds\right|$$
$$\leqq M_q e^{-qu}\int_{-\infty}^{\infty} e^{q\tau}\left|\Phi\left(\tau-\frac{\pi i}{2}\right)-\Phi\left(\tau+\frac{\pi i}{2}\right)\right|d\tau, \tag{27}$$
$$(2m\leqq q<2(m+1)).$$

On using (26) with $p=q+1$, for the part of the integral of the second member of (27) extended over $[0,\ \infty)$, and using the evaluation

$$\left|\Phi\left(\tau-\frac{\pi i}{2}\right)-\Phi\left(\tau+\frac{\pi i}{2}\right)\right|<2M'$$

for the part of the integral extended over $(-\infty,\ 0]$, we find

$$\left|\int_{\Gamma_c}\left[F(s)-\sum_1^m d_n e^{-2ns}\right]\Phi(w-s)ds\right|$$
$$\leqq 2M_q e^{-qu}[M'/q+M_{q+1}'].$$

We find also immediately

$$\left|\int_{c-\pi i/2}^{c+\pi i/2}\left[F(s)-\sum_1^m d_n e^{-2ns}\right]\Phi(w-s)ds\right|\leqq \pi M' M_q e^{-qc}.$$

Therefore for $u<c$:

$$|A| \leqq 2M_q e^{-qu}\left[\left(\frac{\pi}{2}+\frac{1}{q}\right)M' + M'_{q+1}\right],$$

$$(2m \leqq q < 2(m+1)).$$

And similarly

$$|B| \leqq 2M'_q e^{-qu}\left[\left(\frac{\pi}{2}+\frac{1}{q}\right)M + M_{q+1}\right],$$

$$(2m \leqq q < 2(m+1)).$$

If $0<v<\pi$, $\Phi(w-s)$ takes the same values on $t=\pi/2$ as on $t=-(\pi/2)$, and we may thus write:

$$C = \int_{u-c-\pi i/2}^{u-c+\pi i/2}\left[F(s) - \sum_1^m d_n e^{-2ns}\right]$$
$$\cdot\left[\Phi(w-s) - \sum_1^m d'_n e^{-2n(w-s)}\right]ds$$
$$+\int_{u-c}^{c}\left[F\left(\sigma-\frac{\pi i}{2}\right) - F\left(\sigma-\frac{\pi i}{2}\right)\right]$$
$$\cdot\left[\Phi\left(w-\sigma+\frac{\pi i}{2}\right) - \sum_1^m d'_n e^{-2n(w-\sigma+\pi i/2)}\right]d\sigma.$$

On conducting the proofs as above we find

$$|C| \leqq 2M'_q e^{-qu}\left(\frac{M}{q}+\frac{\pi M_q}{2}+M_{q+1}\right).$$

The evaluations of A, B and C, and the fact that $\log M_n$ and $\log M'_n$ are convex functions of n lead, for $u<c$ to an inequality of the form:

$$(28)\quad \left|H_c(w) - \sum_1^m d_n d'_n e^{-2nw}\right|$$
$$\leqq e^{-qu}[C_1 M_q + C_2 M'_q + C_3 M_{q+1} M'_{q+1}],\ (2m \leqq q < 2(m+1)$$

where the C's are constants.

The conclusion of our lemma follows immediately from these

inequalities, if we recall again that $\log M_n$ and $\log M_n'$ are convex functions of n and that there exists a constant $a>0$ such that $M_n \geqq a$, $M_n' \geqq a > 0$.

REMARK. *Without supposing that the sequences* $\{M_n\}$ *and* $\{M_n'\}$ *are bounded away from zero, the other hypotheses of lemma* 10 *being satisfied, we can assert that the inequalities* (28) *hold.*

LEMMA 13. *If to the conditions of lemma* 11 *we add the condition that there exists a positive* δ *such that* $|F(s)| \leqq Me^{-\delta|\sigma|}$, $|\Phi(s)| \leqq M'e^{-\delta|\sigma|}$ *for* $|t| < \pi/2$, *then* $H(w) = \lim_{c=\infty} H_c(w)$ *exists, and is a holomorphic and bounded function for* $0 < v < \pi$. *If, moreover,* $H(w) \equiv 0$, *then either* $F(s) \equiv 0$ *or* $\Phi(s) \equiv 0$.

We know by lemma 11 that $H_c(w)$ is holomorphic in the strip $0 < v < \pi$. We find, from the definition of $H_c(w)$:

$$
\begin{aligned}
|H_c(w)| &\leqq \frac{1}{\pi}\left|\int_{w-(c+\pi i/2)}^{w-(c-i\pi/2)} F(s)\Phi(w-s)ds\right| \\
&\quad + \frac{1}{\pi}\left|\int_{u-c}^{c}\left[F'\left(\sigma+\frac{\pi i}{2}\right) - F\left(\sigma-\frac{\pi i}{2}\right)\right]\right. \\
&\quad \left.\cdot\Phi\left(u-\sigma+i\left(v-\frac{\pi}{2}\right)\right)d\sigma\right| \\
&\leqq MM' + (2MM'/\pi)\int_{-\infty}^{\infty} e^{-\delta|\sigma|}d\sigma = MM'(1+4/\pi\delta).
\end{aligned}
$$

The functions H_c are therefore bounded uniformly with respect to c in the strip $0 < v < \pi$.

If $c' > c$, denote by Γ the rectangle whose vertices are $u-c-(\pi i/2)$, $u-c'-(\pi i/2)$, $u-c'+(\pi i/2)$, $u-c+(\pi i/2)$. We may then write:

$$
\begin{aligned}
\pi i[H_c(w) - H_{c'}(w)] &= \int_\Gamma F(s)\Phi(w-s)ds \\
&\quad + \int_c^{c'}\left[F'\left(\sigma+\frac{\pi i}{2}\right) - F\left(\sigma-\frac{\pi i}{2}\right)\right] \\
&\quad \cdot\Phi\left(u-\sigma+i\left(v-\frac{\pi}{2}\right)\right)d\sigma.
\end{aligned} \tag{29}
$$

But, we have:

$$\int_\Gamma F(s)\Phi(w-s)ds = \int_{u-c'-\pi i/2}^{u-c+\pi i/2} - \int_{u-c-\pi i/2}^{u-c+\pi i/2} F(s)\Phi(w-s)ds$$
$$+ \int_{u-c'}^{u-c}\left[F\left(\sigma+\frac{\pi i}{2}\right)\right] - F\left(\sigma-\frac{\pi i}{2}\right)\Big]$$
$$\cdot\Phi\left(u-\sigma+i\left(v-\frac{\pi}{2}\right)\right)d\sigma;$$

and we find then

$$\left|\int_\Gamma F(s)\Phi(w-s)ds\right| \leqq 2\pi MM'e^{\delta(u-c)} + 2MM'\int_{-\infty}^{u-c} e^{\delta\sigma}d\sigma,$$

which shows that $\int_\Gamma$ tends in any compact contained in $0<v<\pi$ uniformly to zero, as c tends to $+\infty$. It is also easy to prove that the last integral in (29) is smaller, in absolute value, than $2MM'\int_c^\infty e^{-\delta\sigma}d\sigma$. Hence, $\lim_{c=\infty} H_c(w)$ exists, and it is holomorphic, and, as shown before, bounded in $0<v<\pi$.

From

$$H(w) = \lim_{c=\infty}\int_{\Gamma_c(w)} F(s)\Phi(w-s)ds$$
$$= \lim_{c=\infty}\Bigg[\int_{u-c-\pi i/2}^{u-c+\pi i/2} F(s)\Phi(w-s)ds$$
$$+ \int_{u-c}^{c}\left(F\left(\sigma+\frac{\pi i}{2}\right) - F\left(\sigma-\frac{\pi i}{2}\right)\right)$$
$$\cdot\Phi\left(u-\sigma+i\left(v-\frac{\pi}{2}\right)\right)d\sigma\Bigg]$$

and the fact that the first integral of the second member tends to zero follows that

$$H(w) = \frac{1}{\pi i}\int \Psi(\sigma)\Phi\left((u-\sigma)+i\left(v-\frac{\pi}{2}\right)\right)d\sigma,$$

where

$$\Psi(\sigma) = F\left(\sigma + \frac{\pi i}{2}\right) - F\left(\sigma - \frac{\pi i}{2}\right).$$

On using the same idea as that used at the end of the proof of lemma 7, we can easily finish the proof of the present lemma. The function

$$g(w) = \pi i H\left(w + \frac{\pi i}{2}\right)$$

is holomorphic and bounded in the strip $|v| < (\pi/2)$. We have $\Psi(\sigma) = O(e^{-\delta|\sigma|})$, $\Phi(s) = O(e^{-\delta|\sigma|})$ and

$$g(w) = \int \Psi(\sigma)\Phi(w - \sigma)d\sigma,$$

with $\Psi(\sigma) \in L$, $\Phi(\sigma) \in L$. It can also be seen that for $0 < \delta' < \delta$: $g(w) = O(e^{-\delta'|\sigma|})$.

It suffices now to imitate the reasoning of the end of the proof of lemma 7 in order to show that, if $H \equiv 0$, that is to say, if $g \equiv 0$, then either $\Psi(\sigma) \equiv 0$, or $\Phi(\sigma) \equiv 0$. Now, if $\Phi(\sigma) \equiv 0$, then the conclusion of the lemma is established. If $\Psi(\sigma) \equiv 0$, that is

$$F(\sigma + \frac{\pi i}{2}) = F(\sigma - \frac{\pi i}{2}),$$

then $F(s)$ is an entire function, bounded in the plane (since it is periodic), and satisfying the inequality $F(\sigma) = O(e^{-\delta|\sigma|})$; therefore $F \equiv 0$. The lemma is thus proved.

We can now prove the following theorem.

THEOREM 6. *Let $F(s)$ and $\Phi(s)$ be holomorphic in $|t| < (\pi/2)$ and continuous on its boundary. Let Λ, Λ' be two sequences of positive even integers without common terms, the union of the two sequences forming the set of all the even positive integers. Suppose that in $|t| < (\pi/2)$ the inequalities*

$$(30) \quad \left| F(s) - \sum_{1}^{m} a_n e^{-\lambda_n s} \right| \leqq M_q e^{-q\sigma}, \quad (\lambda_m \leqq q < \lambda_{m+1},\ m \geqq 1),$$

$$(31)\qquad \left| F(s) - \sum_{1}^{m} b_n e^{-\lambda'_n s} \right| \leqq M'_q e^{-q\sigma}, \qquad (\lambda'_m \leqq q < \lambda'_{m+1},\ m \geqq 1)$$

hold, $\log M_n$ *and* $\log M'_n$ *being convex functions of* n.

And suppose that there exists a positive δ *such that in* $|t| < \pi/2$

$$\begin{aligned} F(s) &= 0\ (e^{\delta\sigma}) \\ \Phi(s) &= 0\ (e^{\delta\sigma}) \end{aligned} \qquad (\sigma \to -\infty).$$

Suppose, finally, that

$$(32)\qquad \sum \frac{M_n M'_n}{M_{n+1} M'_{n+1}} = \infty.$$

Then, either $F \equiv 0$, *in which case* $a_n = 0 (n \geqq 1)$, *or* $\Phi \equiv 0$, *in which case* $b_n = 0$ $(n \geqq 1)$.

From lemma 13 (and the hypotheses of this theorem) it follows that the function $H(w)$, corresponding to the functions F and Φ of this statement, exists; and, d_n being the coefficient of e^{-2ns} for $F(s)$, d'_n being the coefficient of e^{-2ns} for $\Phi(s)$, ($d_n = 0$, if $2n \in \Lambda'$, and $d'_n = 0$, if $2n \in \Lambda$), $d_n d'_n = 0$ $(n \geqq 1)$.

Suppose first, that $M_n > a$, $M'_n > a > 0$. By lemma 12, we have:

$$|H_c(w)| \leqq K M_{q+1} M'_{q+1} e^{-q\sigma}, \qquad (q \geqq 1,\ u < c,\ 0 < v < \pi).$$

Since $H(w) = \lim_{c=\infty} H_c(w)$, we have in $0 < v < \pi$:

$$(33)\qquad |H(w)| \leqq K M_{q+1} M'_{q+1} e^{-q\sigma}, \qquad (q \geqq 2).$$

If, now, $\liminf_{n=\infty} M_n = 0$, then, by the convexity of $\log M_n$, we have $\lim_{n=\infty} M_n = 0$; there exist then positive constants A and B such that $M_n \leqq AB^n (n \geqq 1)$. If we replace in (32) M_n by $M''_n = AB^n$, the corresponding series will still diverge. If we had $\liminf_{n\to\infty} M'_n = 0$, then M'_n could be replaced by $M_n^* = A_1 B_1^n$, with $M'_n \leqq A_1 B_1^n$. In other words, even if one (or both) of the sequences $\{M_n\}$, $\{M'_n\}$ were not bounded away from zero, (33) would still hold, on replacing, if necessary, $\{M_n\}$, $\{M'_n\}$ by other sequences, still satisfying (32).

It follows then from the solution of Watson's problem [1] that $H \equiv 0$. The conclusion of the theorem follows then from lemma 13.

Theorem 5 can be extended to a set of N functions in the following fashion [8].

THEOREM 7. *If* 1°) $F_1, F_2, \cdots, F_N$ *are holomorphic in* $|t| < (\pi/2)$, *continuous on the boundary*; 2°) *there exists a positive constant* δ *such that* $F_p(s) = O(e^{\delta\sigma})(\sigma \to -\infty)(p = 1, \cdots, N)$; 3°) *in* $|t| < \pi/2$:

$$\left| F_p(s) - \sum_1^m d_n^{(p)} e^{-\lambda_n^{(p)} s} \right| \leqq M_q^{(p)} e^{-qs}, \qquad (\lambda_m^{(p)} \leqq q < \lambda_{m+1}^{(p)},\ m \geqq 1),$$

$\bigcap_{1 \leq p \leq N} \Lambda^{(p)} = void$, $(\Lambda^{(p)} = \{\lambda_n^{(p)}\})$, $\bigcup \Lambda^{(p)} =$ *the set of all the positive even integers*;

4°) $$\sum_{n=1}^{\infty} \prod_{p=1}^{N} \frac{M_n^{(p)}}{M_{n+1}^{(p)}} = \infty :$$

Then, at least one of the functions $F_p(z)$ *is identically zero, and the corresponding coefficients are all zero.*

Indeed, lemmas 11, 12, 13 may be used with an induction process. If F, Φ satisfy the hypotheses of lemma 11, the function

$$g(F, \Phi) = g(w) = H\left(s + \frac{\pi i}{2}\right)$$

satisfies, as we have seen in the proof of lemma 11, the same hypotheses; and, if $g \equiv 0$, then, either $F \equiv 0$, or $\Phi \equiv 0$. Moreover, if F and Φ admit asymptotic representations in $|t| < (\pi/2)$, with coefficients d_n, d_n', and constants M_n and M_n' respectively, then $g(s)$ also admits an asymptotic representation in $|t| < (\pi/2)$, with coefficients $(-i)^n d_n d_n'$ and constants $K M_{n+1} M_{n+1}'$. This shows that the set $F_1, F_2, \cdots, F_N$ can be replaced by $g(F_1, F_2), F_3, \cdots, F_N$; the number of functions is reduced (and the case N is reduced to the case $N-1$). Since it is proved for the case $N = 2$, the theorem is proved in the general case.

4. Coming back to the composition of infinitely differentiable functions, we shall introduce the spectrum of such functions. This permits us to specialize the theorems given above in an interesting way. The spectra are then combined in a way similar to that in which the singularities are in Hadamard's theorem.

Such a specialization is given in Lefkovits' Rice Institute Ph.D. Thesis [9], with somewhat different definitions and more involved proofs.

E_1 and E_2 being two sets of numbers, we shall denote by $E_1 \otimes E_2$ the set of all the numbers of the form $\alpha\beta$, where $\alpha \in E_1$, $\beta \in E_2$.

We will now prove the following theorem:

THEOREM 8. *Let* $\mathfrak{M}$, $\mathfrak{M}'$ *be two sequences of positive numbers,* $\log M_n$ *and* $\log M_n'$ *being convex functions of* n. $(\mathfrak{M} = \{M_n\}, \mathfrak{M}' = \{M_n'\})$. *Let* Λ *and* Λ' *be two increasing sequences of even positive integers and let* $\mathfrak{N}$ *be the sequence complementary to* $\Lambda \cup \Lambda'$, *with respect to the sequence of all even positive integers. Let* E_1, E_2 *be closed, symmetric sets on* R, *neither containing the origin (the point* 0).

If the sequence $\{M_n M_n'\}$ *is associated* (2) *with the set* $E_1 \otimes E_2$ *and the sequence* $\mathfrak{N}$, *then, the only even function contained in one of the classes* $L^2(\mathfrak{M}, \Lambda, E_1)$, $L^2(\mathfrak{M}', \Lambda', E_2)$ *is the function identically zero.*

This theorem generalizes theorem 4, (with $p = 2$). Indeed, if $E_1 = E_2 = R$ then $E_1 \otimes E_2 = R$, and, if (22) is satisfied, $\{M_n M_n'\}$ is associated with R (Δ_R is the upper half-plane) and the sequence complementary to $\Lambda \cup \Lambda'$ (the sequence of all the positive even numbers).

Let us suppose, for instance, that there exists a function $f \in L(\mathfrak{M}, \Lambda, E_1)$, not identically zero. Consider then the function $H(w)$ introduced in lemma 6. By lemma 6 we have:

$$(-i)^{\lambda_n} \int H(w) w^{\lambda_n} dw = \sqrt{2\pi} f^{(\lambda_n)}(0) = 0,$$

and

$$\int_0^\infty |H(w)| w^x dw \leqq CM(x+1), \qquad (x \geqq 0),$$

where $\log M(x)$ denotes the ordinate of the polygonal line $(x \geqq 0)$, with vertices $(n, \log M_n)$.

We note, on the other hand, that

$$H(w) = 0 \text{ p.p. on } \sigma(f).$$

Let us also set $K(x)=H(e^x)e^x$, $s(f)=\{x|\, e^x\in\sigma(f)\}$. We have $K(x)=0$ p.p. if $x\notin s(f)$, and

$$\int K(t)e^{\lambda_n t}dt = 0, \tag{34}$$

$$\int |\, K(t)\,|e^{tz}dt \leqq CM(x+1), \qquad (x\geqq 0). \tag{35}$$

Let now $g\in L^2(\mathfrak{M}', \Lambda', E_2)$, and let $G(z)=(G^+, G^-)=\mathfrak{FC}(g)$.

We have, by lemma 3, for $|y|>0$

$$\left|\sqrt{2\pi}G(z) - \sum_{n=1}^{m}\frac{g^{(2n)}(0)}{(iz)^{2n+1}}\right| \leqq AM'_{q+1}|\, z\,|^{-q-1}|\, y\,|^{-1/2}, \qquad (2m\leqq q<2m+1), \tag{36}$$

where A is a positive constant, the function $G(z)$ being holomorphic in $\Delta_{\sigma(g)}$. From the fact that g is even it follows that $G(z)$ is odd. On setting $\Phi(s)=\sqrt{2\pi}e^s G(-ie^s)$, the function $\Phi(s)$ is holomorphic and uniform in the part $\mathfrak{D}$ of the plane s obtained when (on setting $s(g)=\{x|\, e^x\in\sigma(g)\}$) all the points $\sigma+(2k+1)\cdot(\pi i/2)$, $\sigma\in s(g)$, k any integer, are removed. We have, moreover, in $\mathfrak{D}$: $\Phi(s+\pi i)=\Phi(s)$.

Since the origin does not belong to $\sigma(g)$, $\mathfrak{D}$ contains a half-plane P_{σ_0} given by $\sigma<\sigma_0$.

We have in $\mathfrak{D}$:

$$\left|\Phi(s) - \sum_{n=1}^{m} g^{(2n)}(0)e^{-2ns}\right| \leqq AM'_{q+1}e^{-q\sigma}(e^{\sigma}\,|\cos t|\,)^{-1/2}, \quad (2m\leqq q<2(m+1)), \tag{37}$$

and, since $g(0)=0$, (37) allows us also to write, for $s\in\mathfrak{D}$:

$$|\,\Phi(s)\,| \leqq AM_2 e^{-\sigma}(e^{\sigma}\,|\cos t)\,|\,)^{-1/2}. \tag{38}$$

We now consider the integral

$$\Psi(s) = \int \Phi(s-x)K(x)dx, \tag{39}$$

which converges for s real; this follows from (35) and (38). Since

$K(x)=0$ p.p. on $\mathcal{C}s(f)$ ($\mathcal{C}E$ meaning the complementary set of E with respect to R), we may write

$$\Psi(s) = \int_{s(f)} \Phi(s-x)K(x)dx. \tag{40}$$

It follows from (38) that the integrand in (40) is, in absolute value, not larger than $A_1M_2|\cos t|^{-1/2}e^{-3\sigma/2}|K(x)|e^{3x/2}$. Therefore, by (35), (40) converges uniformly in each compact contained in $|t|<(\pi/2)$; $\Psi(s)$ is a holomorphic function in $|t|<(\pi/2)$ and satisfies there the inequality:

$$|\Psi(s)| < AM_2|\cos t|^{-1/2}e^{-3\sigma/2}\int |K(x)|e^{3x/2}dx$$
$$\leqq AM_2M\left(\frac{5}{2}\right)e^{3\sigma/2}|\cos t|^{-1/2} = Be^{-3\sigma/2}|\cos t|^{-1/2}.$$

On the other hand, if $\sigma\notin s_{f,g}=s(f)\oplus s(g)$ [i.e. $e^{\sigma}\notin\sigma(f)\otimes\sigma(g)$],[9] and if $x\in s(f)$, then $\sigma-x\notin s(g)$. In a general way, if $[a, b]\subset\mathcal{C}s_{f,g}$, then there exists a positive d such that for any $\sigma\in[a, b]$ and any $\gamma\in s_{f,g}$, $|\sigma-\gamma|\geqq d$. Indeed, $\delta>0$ being any, if $|\sigma-\alpha-\beta|<\delta$, $\sigma\in[a, b]$, $\alpha\in s(f)$, $\beta\in s(g)$, then necessarily $\alpha<b+\delta-\sigma_0$, $\beta<b+\delta-\sigma_0$, $\sigma_0<\alpha$, $\sigma_0<\beta$. In other words, the points γ of $s_{f,g}$ such that there exists a $\sigma\in[a, b]$ with $|\sigma-\gamma|<\delta$ form a compact, from which our assertion follows.

It can now be proved that when $s\in S[a, b]$, where $S[a, b]$ is the square centered at $[(a+b)+\pi i]/2$, with sides parallel to the axes, with $[a, b]\subset\mathcal{C}s_{f,g}$, $x\in s(f)$, then $\Phi(s-x)$ is holomorphic (as a function of s), and there exists a positive constant A such that in $S[a, b]$: $|\Phi(s-x)|\leqq Ae^{(5/2)x}$. Indeed, by what we have just seen, there exists a $d>0$ such that, for every $\beta\in s(g)$: $|(s-x)-\beta|\geqq d$; therefore $\Phi(s-x)$ is holomorphic in $S[a, b]$; $s-x$ is on the square S_x obtained by translation $-x$ of $S[a, b]$. When $s-x$ belongs to the square S_x^* concentric with S_x and with sides equal to $(b-a)+d/2$, $\Phi(s-x)$ is holomorphic and satisfies, by (38), the inequality $|\Phi(s-x)|\leqq B|\cos t|^{-1/2}e^{(3/2)(\sigma-x)}\leqq B_1|t-\pi/2|e^{(3/2)x}$. On using the same device as that used in the proof of lemma 2 (here $|t-\pi/2|$ plays the same role as $|y|$ in lemma 2), we arrive at the desired conclusion. It follows

then from (35) that, for $s \in S[a, b]$):

$$\int |\Phi(s-x)K(x)|\,dx \leqq A \int e^{(3/2)x} |K(x)|\,dx < \infty.$$

The formula (40) shows then that $\Psi(s)$ is holomorphic in $S[a, b]$. We may therefore assert that $\Psi(s)$ is a holomorphic and uniform function in the domain $\mathfrak{D}^*$ obtained on removing from the plane the points $\sigma+(2k+1)\pi i/2$, where $\sigma \in \mathfrak{C}s_{f,g} = \mathfrak{C}[s(f) \oplus s(g)]$; and, it follows from (38) and (35) that, for $t \neq (2k+1)\pi/2$:

$$|\Psi(s)| < Le^{-\sigma}(e^{\sigma}|\cos t|)^{-1/2}. \tag{41}$$

By (37) and (35) we may write:

$$\left|\int \left[\Phi(s-x) - \sum_{n=1}^{m} g^{(2n)}(0)e^{-2n(s-x)}\right] K(x)dx\right|$$

$$= \left|\Psi(s) - \sum_{n=1}^{m} g^{(2n)}(0)e^{-2ns} \int K(x)e^{2nx}dx\right| \tag{42}$$

$$\leqq A_1 M'_{q+1} M_{q+1} e^{-q\sigma}(e^{\sigma}|\cos t|)^{-1/2},$$

$$(2m \leqq q < 2(m+1),\ m \geqq 1).$$

Since $g^{(\mu_n)}(0) = 0$, it follows from (42) and (34) that, on setting

$$d_n = g^{(\nu_n)}(0) \int K(x)e^{\nu_n x}dx,$$

(where, we recall, $\mathfrak{N} = \{\nu_n\}$ is the sequence of positive even integers complementary to $\Lambda \cup \Lambda'$ with respect to the set of all the positive even integers) the following inequalities hold for $|t| < (\pi/2)$:

$$\left|\Psi(s) - \sum_{1}^{m} d_n e^{-\nu_n s}\right|$$

$$\leqq A M_{q+1} M_{q+1} e^{-q\sigma}(e^{\sigma}|\cos t|)^{-1/2}, \qquad (\nu_m \leqq q < \nu_{m+1},\ m \geqq 1).$$

Coming back to the variable $z = -ie^s$ we see that the function $\Psi_1(z) = e^{-s}\Psi(s)$ is holomorphic and uniform in the plane z, with the points $\sigma(f) \oplus \sigma(\phi)$ removed; and for $|y| > 0$ the inequalities

$$\left|\Psi_1(z) - \sum_1^m \frac{i^n d_n}{g^{\nu_n+1}}\right|$$

$$\leqq A M_{q+1} M'_{q+1} |z|^{-q-1} |y|^{-1/2}, \quad (\nu_m \leqq q < \nu_{m+1}, m \geqq 1),$$

and

$$|\Psi_1(z)| \leqq L|y|^{-1/2}$$

hold. ($\Psi_1(z)$ is regular also at the origin by (41), and the last inequality holds by (41)).

Since the sequence $\{M_n M'_n\}$ is associated (2) with the set $E_1 \oplus E_2$, thus, a fortiori, with $\sigma(f) \oplus \sigma(\phi)$, we conclude that $\Psi_1(z) \equiv 0$, that is to say $\Psi(s) \equiv 0$. We have therefore proved that

$$\int \Phi(s-x)K(x)dx \equiv 0. \tag{43}$$

Consider now the function $e^{\lambda x}\Phi(x)$, with $3/2 < \lambda < 1$, x real. We have, by (38) and (37):

$$|e^{\lambda x}\Phi(x)| \leqq C_1 e^{(\lambda - 3/2)x},$$

$$|e^{\lambda x}\Phi(x) - g^{(l)}(0)e^{-(l-\lambda)x}| < A_1 M'_{l+1} e^{-(l-\lambda+1/2)x},$$

where l is the smallest positive even integer not a $\lambda'_n (n \geqq 1)$. Since $l \geqq 2$, we see that $e^{\lambda x}\Phi(x) \in L$; we know also that $e^{\lambda x}K(x) \in L$. In copying now the proof of lemma 7, we find then, as a consequence of (43) (since $K(x) \not\equiv 0$ (p.p.)) that $\Phi(x) \equiv 0$; therefore $G(x) \equiv 0$, that is to say $g \equiv 0$. This proves our theorem.

Of course, theorem 8 can be generalized to $N \geqq 2$ classes of functions $L^2(\mathfrak{M}, \Lambda, E)$. By $\prod_{1 \leq n \leq N} \otimes E_N = E_1 \otimes E_2 \otimes \cdots \otimes E_N$ we understand the composite product of the sets $E_1, \cdots, E_N$, where $E_1 \otimes E_2 \cdots \otimes E_N = (E_1 \otimes \cdots \otimes E_{N-1}) \otimes E_N$ (this product is commutative and associative) [9].

THEOREM 9. *Let $\mathfrak{M}^{(j)}$ $(j = 1, \cdots, N)$ be N sequences of positive numbers,* $\log M_n^{(j)} (1 \leqq j \leqq N)$ *being a convex function of n. Let $\Lambda^{(j)} (1 \leqq j \leqq N)$ be N increasing sequences of even positive numbers, and let $\mathfrak{N}$ be the sequence complementary to $\bigcup_{1 \leq j \leq n} \Lambda^{(j)}$ with respect to the sequence of all even positive integers. Let $E^j (1 \leqq j \leqq N)$ be*

closed symmetric sets on R, none of them containing the origin (the point 0).

If the sequence $\{\prod_{1\leqq j\leqq N} M_n^{(j)}\}$ is associated (2) with the set $\prod_{1\leqq j\leqq N}\otimes E_j$ and the sequence $\mathfrak{M}$, then the only even function contained in one of the classes. $L^2(\mathfrak{M}^{(j)}, \Lambda^{(j)}, E_j)(1\leqq j\leqq N)$ is the function identically zero.

The proof of this theorem is based on theorem 8 in the same manner as the proof of theorem 7 is based on theorem 6.

5. Let us give now some applications of the established composition theorems. We shall denote by $L^{(p)}(\mathfrak{M})(\mathfrak{M}=\{M_n\})$ the class of infinitely differentiable functions on R, with $f^{(n)}\in L^p(n\geqq 0)$, $\|f^{(n)}\|_p\leqq M_n(n\geqq 1)$.

THEOREM 10. *Let* log M_n *be a convex function of n such that*

$$\sum\left(\frac{M_n}{M_{n+1}}\right)^2=\infty.$$

If f is an even function belonging to $L^{(\infty)}(\mathfrak{M})$, and if there exists a real constant a such that $f(0)=f(a)=0, f^{(2n)}(0)f^{(2n)}(a)=0$ $(n\geqq 1)$, then f is periodic with period $4a$.

Let Λ be the set of positive even numbers such that $f^{(\lambda_n)}(0)=0$ $(n\geqq 1, \lambda_n\in\Lambda)$, and let Λ' be the sequence complementary to Λ with respect to the set of all the even positive integers. Then $f^{(\lambda_n')}(a)=0$ $(n\geqq 1,\ \lambda_n'\in\Lambda')$. We have obviously: $f(x)\in L^\infty(\mathfrak{M}, \Lambda)$, $(1/2)[f(x+a)+f(-x+a)]\in L^\infty(\mathfrak{M}, \Lambda')$. Therefore, by theorem 4, either $f\equiv 0$, and the theorem is proved; or $f(x+a)+f(-x+a)\equiv 0$, that is to say, $f(x+2a)=f[(x+a)+a]=-f[(-x-a)+a]=-f(-x)=-f(x)=f(x-2a)$. If in the statement of this theorem, $L^{(\infty)}$ is replaced by $L^{(2)}$, then the conclusion is obviously $f\equiv 0$. Since, we conclude, as before, that f is periodic, thus identically zero. All the other composition theorems on classes $L^\infty(\mathfrak{M}, \Lambda, E)$ lead to similar theorems. For instance, theorem 5 leads to:

THEOREM 11. *Let N be a positive integer, let* log M_n *be a convex function of n, and let*

$$\sum\left(\frac{M_n}{M_{n+1}}\right)^N=\infty.$$

If f is an even function belonging to $L^{(\infty)}(\mathfrak{M})$ and if there exist N real constants a_k $(1 \leqq k \leqq N)$ such that

$$f(a_1) = f(a_2) = \cdots = f(a_N) = \prod_{1 \leqq k \leqq N} f^{(2n)}(a_k) = 0, \quad (n \geqq 1),$$

then $f(x)$ is periodic and admits as a period one of the quantities $4a_k$ $(1 \leqq k \leqq N)$. Again, if $L^{(\infty)}$ is replaced by $L^{(2)}$ the conclusion is that $f \equiv 0$.

We have only to apply theorem 5 to the functions

$$\phi_k(x) = \frac{f(x + a_k) + f(-x + a_k)}{2}, \quad (1 \leqq k \leqq N).$$

Under a different form this theorem can be found in [9].

The most interesting theorem of this kind seems to be the following one.

THEOREM 12. *Let $\mathfrak{M}$ be a sequence of positive numbers,* $\log M_n$ *being a convex function of n; let Λ and Λ' be two sequences of even positive integers, and let $\mathfrak{N}$ be the sequence of even positive integers complementary to $\Lambda \cup \Lambda'$ with respect to the set of all the even positive integers. Let E be a closed set on R.*

If f is an even function belonging to $L^{(2)}(\mathfrak{M})$, if $\sigma(f) \in E$, if for two real numbers a_1 and a_2 we have $f(a_1) = f^{(\lambda_n)}(a_1) = 0$ $(\lambda_n \in \Lambda)$, $f(a_2) = f^{(\lambda_n')}(a_2) = 0$ $(\lambda_n' \in \Lambda')$, and if $\mathfrak{M}^2(\mathfrak{M}^2 = \{M_n^2\})$ is associated (2) with $E^2(E^2 = E \otimes E)$ and $\mathfrak{N}$, then f is identically zero.

On considering again the functions $\phi_k = (1/2)[f(x+a_k) + f(-x+a_k)]$ $(k=1, 2)$ and on noticing that $\sigma(\phi_k) = \sigma(f)$, it is sufficient to apply theorem 8, with $L^2(\mathfrak{M}, \Lambda, E)$ and $L^2(\mathfrak{M}, \Lambda', E)$ to have our result (we have of course to continue the proof on the same lines as that of theorem 10). Theorem 9 leads to a similar result where the derivatives of f in N points are involved.

6. Let us now give some deeper applications of composition theorems, already proved in [1]. We need first the following lemma [1].

LEMMA 14. *Let Λ be an infinite sequence of positive increasing integers, and let $\mathfrak{N}$ be the complementary set of Λ with respect to the set of all the positive integers.*

There exists an infinitely differentiable, bounded function, f, on $[0, \infty)$ *satisfying the conditions:*

$$f(0) = f^{(\mu_n)}(0) = 0(\mu_n \in \mathfrak{N}), \qquad f(x) \not\equiv 0,$$

$$\left| f^{(n)}(x) \right| \leqq \lambda_1 \lambda_2 \cdots \lambda_n, \qquad (n \geqq 1)(\lambda_n \in \Lambda).$$

We may suppose that $\mathfrak{N}$ contains at least one term, since otherwise, the existence of f is obvious. Let $\mu_k \in \mathfrak{N}$ be such that $\mu_k + 1 \in \Lambda$ and let $\lambda_m = \mu_k + 1$.

Let $\alpha(z)$ $(z = x + iy)$ realize the conformal mapping of the domain D, defined by removing from the complex plane the ray $x \leqq -1$, on $|\alpha| < 1$, with $\alpha(0) = 0$,[10] and let us set for $u > -1$, $t > -1$:

$$w_1(z; u, t) = \alpha^{(\mu_k)}(u)(1 + u)^{\mu_k} \alpha[(1 + t)z + t],$$

$$w_0(z; u, t) = w_1(z; u, t) - w_1(z; t, u).$$

We have

$$\left. \frac{\partial^{\mu_k} w_0(z; u, t)}{\partial z^{\mu_k}} \right|_{z=0} = 0,$$

for every $u > -1$, $t > -1$. And there exists a couple $u = u_1$, $t = t_1$ such that

$$\left. \frac{\partial^{\mu_k + 1} w_0(z; u_1, t_1)}{\partial z^{\mu_k + 1}} \right|_{z=0} \neq 0,$$

since, otherwise, we would have identically $\alpha^{(\mu_k)}(u) = C(1+u)^a$, where a and C are constants, which is impossible.

With a suitable choice of the positive constant C, the function

$$w(z) = C w_0(z; u_1, t_1)$$

is holomorphic in D, it satisfies there the inequality $|w(z)| \leqq 1$, and

$$w(0) = w^{(\mu_k)}(0) = 0, \qquad w^{(\mu_k + 1)}(0) \neq 0.$$

Let us now set $F(s) = w(e^s)$. $F(s)$ is holomorphic and bounded in the domain Δ, given by the union of the half-plane $\sigma < 0$ and the strip $|t| < \pi$. Let us also set

$$M(r) = \prod_{n \geqq 1, n \neq k} \left(1 + \frac{r^2}{\mu_n^2}\right) = \sum_0^\infty l_n r^{2n}.^{11}$$

We shall prove that the expression

$$\phi(x) = \sum_{n=0}^{\infty} (-1)^n l_n F^{(2n)}(\sigma), \tag{44}$$

$e^{\sigma} = x$, represents an infinitely differentiable function on $[0, \infty)$, and that the function f, we are looking for, is given by $\phi(ax)$, where a is a suitably chosen constant. We have to evaluate, on one hand, the constants l_n by means of the sequence Λ, and, on the other hand, $F^{(2n)}(\sigma)$ on $\sigma \in R$. (44) will give the values of $\phi(x)$ for $x > 0$. For $x = 0$, it will be defined by continuity.

On setting

$$Q(r) = \prod_{n \geq 1} \left(1 + \frac{r^2}{\lambda_n^2}\right),$$

we have

$$M(r) = |\sin i\pi r| \left[\left(1 + \frac{r^2}{\mu_k^2}\right) Q(r)\right]^{-1},$$

$$\log M(r) \leqq \pi r - \log\left(1 + \frac{r^2}{\mu_k^2}\right) - \log Q(r) - \log(\pi r).$$

And, since

$$\log Q(r) = \sum_{n=1}^{\infty} \log\left(1 + \frac{r^2}{\lambda_n^2}\right) = 2r^2 \int_0^{\infty} \frac{N(x)}{x} \frac{dx}{x^2 + r^2}$$

$$\geqq \int_1^r \frac{N(x)}{x}\, dx = N_1(r),$$

where $N(x)$ is the number of λ_n smaller than x, and $l_n \leqq M(r) r^{-2n}$ for every $r > 0$, we may write:

$$\log l_n \leqq \pi r - 3 \log(\pi r) - N_1(r) - 2n \log r + A,$$

where A is a constant. If we set in this inequality $\pi r = 2n$, we get

$$0 \leqq l_n \leqq A_1 (\pi e)^{2n} (2n)^{-2n-3} e^{-N_1(2n/\pi)}$$

$$\leqq A_2 n^{-5/2} \frac{\pi^{2n}}{(2n)!}\, e^{-N_1(2n/\pi)}, \qquad (n \geqq 1). \tag{45}$$

As to the evaluation of $F^{(2n)}(\sigma)$, and its derivatives, with respect to $x(x=e^{\sigma})$, we proceed in the following way. For $0 \leqq \gamma < \pi$, and σ_1 real, any, we have for $|s-\sigma_1| \leqq \gamma$ the inequality (by Cauchy's theorem):

$$|F^{(n)}(s)| \leqq \frac{n!}{(\pi-\gamma)^n} \tag{46}$$

(this is true on $|s-\sigma_1| = \gamma$, therefore also inside this circle). For $\sigma \leqq -\pi$, we may write:

$$|F^{(n)}(s)| \leqq \frac{n!}{\pi^n}. \tag{47}$$

There exists a $\delta = \delta(\sigma_0) > 0$, such that, for every $0 < \gamma < \pi$, and every $\sigma_1 \geqq \sigma_0$, the inequalities $|t| < \pi$, $|e^s - e^{\sigma_1}| \leqq \delta\gamma$ imply $|s-\sigma_1| < \gamma$. On setting, therefore, $x_1 = e^{\sigma_1} \geqq e^{\sigma_0}$ (we have set $z = e^s$), we see that, if $|z-x_1| \leqq \delta\gamma$, we have, for $|t| < \pi$: $|s-\sigma_1| \leqq \gamma$. In this circle $F^{(n)}(s)$ is a holomorphic function of z; we shall denote this function by $w_n(z)$. On $|z-x_1| \leqq \delta\gamma$, we have therefore, by (46):

$$|w_n(z)| \leqq \frac{n!}{(\pi-\gamma)^n}.$$

On setting

$$D_p^{(n)} F(\sigma_1) = \frac{d^p F^{(n)}(\sigma_1)}{dx_1^p} = \frac{d^p w_n(x_1)}{dx_1^p},$$

we may write, for $\sigma_1 \geqq \sigma_0$, $\delta = \delta(\sigma_0)$:

$$|D_p^{(n)} F(\sigma_1)| \leqq \frac{p!}{2\pi}\left|\int_{|z-x_1|=\delta\gamma} \frac{w_n(z)dz}{(z-x_1)^{p+1}}\right| \leqq \frac{p!n!}{(\delta\gamma)^p(\pi-\gamma)^n}. \tag{48}$$

Since $zw_{n-1}'(z) = w_n(z)$ $(n \geqq 1)$, with $w_0(z) = w(z)$, $w_n(z)$ is holomorphic in $|z| < 1$. If $\sigma' < -\pi$, we have, by (47), for $|z| = e^{\sigma'}$:

$$\left|\frac{d^p w_n(z)}{dz^p}\right| = |D_p^{(n)} F(s)| = \frac{p!}{2\pi}\left|\int_{|\zeta-z|=e^{-\pi}-e^{\sigma'}} \frac{w_n(\zeta)d\zeta}{(\zeta-z)^{p+1}}\right| \tag{49}$$

$$\leqq \frac{p!n!}{\pi^n(e^{-\pi}-e^{\sigma'})^p}.$$

This inequality holds also for $|z| \leqq e^{\sigma'}$, that is to say, for $\sigma \leqq \sigma'$. If we set in (48) $\gamma = p\pi(n+p)^{-1}$, $(n \geqq 1,\ p \geqq 1)$, we get, for $\sigma_1 \geqq \sigma_0$:

$$(50) \qquad |D_p^{(n)} F(\sigma_1)| \leqq \frac{p!n!(n+p)^{n+p}}{\pi^{n+p} p^p n^n \delta^p} \leqq c^p n!(n+p)^p \pi^{-n}.$$

From (49), written for $\sigma \leqq \sigma'$, and (50), written for $\sigma_1 \geqq \sigma_0$, and choosing $\sigma_0 < \sigma'$, we get, for every real σ:

$$(51) \qquad |D_p^{(n)} F(\sigma)| \leqq B^p \pi^{-n} n! \max(n^p, p^p),$$

where B is a constant. Coming back now to the definition of ϕ (see (44)), we may write, at least, for $x > 0$:

$$\phi^{(p)}(x) = \sum_{n=0}^{\infty} (-1)^n l_n D_p^{(2n)} F(\sigma), \qquad (p \geqq 0).$$

These series converge for every p, uniformly for $\sigma > -\infty$. We see, indeed, by (45) and (51) that, for $p \geqq 1$:

$$(52) \qquad \begin{aligned} |\phi^{(p)}(x)| &\leqq \sum_{n=0}^{\infty} l_n |D_p^{(2n)} F(\sigma)| \\ &\leqq A_1 B^p \sum_{n=1}^{\infty} [(2n)^p + p^p] n^{-5/2} e^{-N_1(2n/\pi)} + C^p p!. \end{aligned}$$

The expression $C^p p!$ arises from the evaluation of

$$\left| \frac{d^p F(\sigma)}{d(e^\sigma)^p} \right| = \left| \frac{d^p w(x)}{dx^p} \right|,$$

$(x \geqq 0)(|w(z)| \leqq 1$ for $z \in D)$. We have also, by (46) with $\gamma = 0$:

$$(53) \qquad |\phi(x)| \leqq A_1 \sum_{1}^{\infty} n^{-5/2} e^{-N_1(2n/\pi)} + 1.$$

Since the sequence Λ has infinitely many terms $\lim_{x=\infty} N(x) = \infty$; therefore:

$$N_1(r) > \int_{r^{1/2}}^{r} \frac{N(x)}{x}\, dx \geqq \frac{1}{2} N(r^{1/2}) \log r,$$

and $\lim_{r=\infty} N_1(r)(\log r)^{-1} = \infty$. In other words, for every $p \geqq 0$, we have

$$n^p e^{-N_1(2n/\pi)} = o(1).$$

The right hand expressions in (52) and (53) converge therefore, and we have

$$|\phi^{(p)}(x)| \leqq KR^p \left\{ \max_{n \geqq 1} \left[\left(\frac{2n}{\pi} \right)^p e^{-N_1(2n/\pi)} \right] \sum_{n=1}^{\infty} n^{-5/2} + Gp! \right\}, \quad (p \geqq 0),$$

where K, R, G are finite positive constants.

On setting

$$N_2(\sigma) = N_1(e^\sigma) = \int_0^\sigma N(e^u)du,$$

the last inequality permits us to write:

$$|\phi^{(p)}(x)| \leqq M^p \left[\exp \left\{ \max_{\sigma \geqq 0} [p\sigma - N_2(\sigma)] \right\} + p! \right],$$

where M is a constant. Since the λ_n are integers, $N(\lambda) \leqq \lambda$, that is to say, $N_2(\sigma) \leqq e^\sigma$. Therefore: $\max_{\sigma \geqq 0} [p\sigma - N_2(\sigma)] \geqq \log p! + \beta p$, ($\beta$ constant), and hence

$$|\phi^{(p)}(x)| \leqq N^p \exp \left\{ \max_{\sigma \geqq 0} [p\sigma - N_2(\sigma)] \right\}.$$

But $N(e^u)$ is the number of quantities $\log \lambda_n < u$, hence, if λ_q is the largest λ_n smaller than σ:

$$\max_{\sigma \geqq 0} [p\sigma - N_2(\sigma)] = \max_{\sigma \geqq 0} \left[p\sigma - \int_0^\sigma N(e^u)du \right]$$

$$= \max_{\sigma \geqq 0} \{ p[\log \lambda_1 + (\log \lambda_2 - \log \lambda_1) + \cdots + (\log \lambda_q - \log \lambda_{q-1})]$$

$$+ p(\sigma - \log \lambda_q) - [(\log \lambda_2 - \log \lambda_1) + 2(\log \lambda_3 - \log \lambda_2) + \cdots$$

$$+ (q-1)(\log \lambda_q - \log \lambda_{q-1}) + q(\sigma - \log \lambda_q)] \}.$$

The expression of which we take the maximum increases when $q \leqq p$, and decreases when $q \geqq p$. Thus:

$$\max_{\sigma \geqq 0} [p\sigma - N_2(\sigma)] = \log \lambda_1 + \log \lambda_2 + \cdots + \log \lambda_p.$$

We have therefore proved that

$$|\phi^{(p)}(x)| \leqq N^p \lambda_1 \lambda_2 \cdots \lambda_p, \qquad (p \geqq 1).$$

Let us now prove that $\phi(0) = \phi^{(\mu_n)}(0) = 0$, $(n \geqq 1)$. Let us set, for $|z| < 1$: $w(z) = \sum_{n=1}^{\infty} a_n z^n$,[12] and let $F_1(s) = \sum |a_n| e^{ns}$. For $\sigma < \sigma_2 < -\pi$, we have $|F_1^{(n)}(s)| \leqq An!\, \pi^{-n}$, where A is a constant. Hence, for $\sigma < \sigma_2$:

$$0 < F_1^{(2n)}(\sigma) = \sum_{p=1}^{\infty} |a_p| p^{2n} e^{p\sigma} \leqq A \frac{(2n)!}{\pi^{2n}}.$$

From (45) it follows then that the double series

$$\sum_{n=0}^{\infty} l_n F_1^{(2n)}(\sigma) = \sum_{n=0}^{\infty} l_n \sum_{p=1}^{\infty} |a_p| p^{2n} e^{p\sigma}$$

converges for $\sigma < \sigma_2$. Thus, for $0 \leqq x < e^{\sigma_2}$, we may write:

$$\phi(x) = \sum_{n=0}^{\infty} (-1)^n l_n \sum_{p=1}^{\infty} a_p p^{2n} e^{p\sigma} = \sum_{p=1}^{\infty} a_p e^{p\sigma} \sum_{n=0}^{\infty} (-1)^n l_n p^{2n}$$

$$= \sum_{p=1}^{\infty} a_p M(ip) e^{p\sigma} = \sum_{p=1}^{\infty} a_p M(ip) x^p,$$

(with $\phi(0)$ being defined as being equal to zero, by continuity). But $a_{\mu_k} = w^{(\mu_k)}(0)(\mu_k!)^{-1} = 0$, and $M(i\mu_n) = 0$, for $n \neq k$, thus

$$\phi(x) = \sum_{n=1}^{\infty} a_{\lambda_n} M(i\lambda_n) x^{\lambda_n}.$$

Since

$$a_{\lambda_m} = w^{(\lambda_m)}(0)(\lambda_m!)^{-1} = w^{(\mu_k+1)}(0)[(\mu_k + 1)!]^{-1},$$

with $w^{(\mu_k+1)}(0) \neq 0$, we see that $\phi(x)$ is not identically zero. The function $f(x) = \phi(x/N)$ satisfies all the desired conditions and our lemma is proved.

LEMMA 15. *If Λ is an infinite sequence of even positive integers,*

and if $\mathfrak{N}$ is the complementary set of Λ with respect to all the positive integers, there exists an even infinitely differentiable, bounded function f, on R, satisfying on R the conditions

$$f(0) = f^{(\mu_n)}(0) = 0, \qquad (\mu_n \in \mathfrak{N}), f \not\equiv 0,$$
$$|f^{(n)}(x)| \leqq \lambda_1\lambda_2 \cdots \lambda_n, \qquad (n \geqq 1)(\lambda_n \in \Lambda).$$

This follows immediately from lemma 14. It is sufficient to continue by symmetry on $(-\infty, 0]$ the function f of lemma 14.

Theorem 4 together with lemma 15 leads immediately to the following theorem (see [1]).

THEOREM 13. *Let Λ be a sequence of even positive integers, and let* log M_n *be a convex function of n, $(M_n > 0, n \geqq 1)$. If*

$$\sum \frac{M_n}{M_{n+1}} \cdot \frac{1}{\lambda_n} = \infty$$

the only even function belonging to $L^\infty(\mathfrak{M}, \Lambda)$ is the function identically zero.

From this theorem follows:

THEOREM 14. *If the sequence of even positive integers Λ is such that*

$$\sum \frac{1}{\lambda_n} = \infty, \tag{54}$$

the only even, bonded, infinitely differentiable function on R with

$$|f^{(n)}(x)| \leqq 1, \qquad (n \geqq 1)$$
$$f(0) = f^{(\lambda_n)}(0) = 0, \qquad (\lambda_n \in \Lambda),$$

is the function $f \equiv 0$.

The condition (54) is not only sufficient for the conclusion of theorem 14, but also necessary. The following theorem holds, indeed:

THEOREM 15. *If the sequence of positive integers Λ is such that*

$$\sum \frac{1}{\lambda_n} < \infty, \tag{55}$$

then there exists an infinitely differentiable function $f(x)$ on R (not identically zero), with

$$(56)\qquad \begin{aligned} &f(0) = f^{(\lambda_n)}(0) = 0\ (\lambda_n \in \Lambda), f^{(m)}(0) \neq 0 \text{ for } m \notin \Lambda. \\ &\left| f^{(n)}(x) \right| \leqq 1\ (n \geqq 0). \end{aligned}$$

For the proof of this theorem, we send the reader to our book, mentioned above [1].

THEOREM 16. *If the sequence* Λ *is composed of even integers, and if* (55) *holds, then there exists an even, infinitely differentiable function (not identically zero) with*

$$f(0) = f^{(\lambda_n)}(0) = 0,\ (\lambda_n \in \Lambda), f^{(2p)}(0) \neq 0,\ (2p \notin \Lambda,\ p \neq 0),$$

$$\left| f^{(n)}(x) \right| \leqq 1.$$

Theorem 16 follows immediately from theorem 15; the function $[f(x)+f(-x)]/2$, where f satisfies the conditions (56), with λ_n even, satisfies the conditions of theorem 16.

7. The essential hypothesis in many of the proved theorems was the fact that "a sequence $\mathfrak{M}$ was associated with a closed set E and a sequence of integers Λ." This hypothesis was expressed on page 6. Our purpose now is to express explicitly, by an analytic relationship, such an association between the elements $\mathfrak{M}$, E and Λ. If $E=R$, relationships of the form (9), furnish a solution of the problem. In order to have a general expression, especially when Λ is not the set of all the positive integers, one has to apply the adherence theorem, that is to say, theorem 1. The next theorem gives general conditions for an association of the desired kind, even if they are far from the best possible conditions.

THEOREM 17. *Let* $\mathfrak{M}$ *be a sequence of positive numbers,* E *a closed, symmetric set on* R, *and* Λ *a sequence of positive integers. The sequence* $\mathfrak{M}$ *is associated* (p) $(p \geqq 1$ *any) with* E *and* Λ, *if the following conditions are satisfied:*

$D^*(\Lambda)$ *being the upper density function of* Λ, *and* D^* *the upper density of this sequence:* $D^* < 1/2$, *and there exists a continuous function* $V(x) \geqq 1/2$, *of bounded variation on* $[0, \infty)$, *such that* $V(x) = 1/2$ *for* $x \in E$, *and such that, on setting*

$$C(\sigma) = \operatorname*{Sup}_{n \geqq 1} (n\sigma - \log M_n),$$

the relationship

$$(57)\qquad \int^{\infty} C(\sigma)\exp\left[-\frac{1}{2}\int^{\sigma}\frac{du}{V(e^u)-D[C(u)]}\right]d\sigma=\infty$$

is satisfied [10].

Let $\Phi(z)$ be a function holomorphic in Δ_E (see page 5 for the definition of Δ_E), and let it satisfy there the inequalities (5). Let $0<a<1$, small enough, and let us apply to the set E_a lemma 2. (As a matter of fact, we suppose that $E\neq R$, and that a is small enough in order that $\mathcal{C}E_a$ not be empty. If we have $E=R$ the proof becomes only simpler, and many of the details which follow are superfluous.) Let us denote by Π_a the part of the z-plane composed of all the points not belonging to the following three sets: 1°) the union of squares with centers at ξ, $\xi\in E$, $\xi\geqq a$, and sides of length $2a$, parallel to the axes; 2°) the set symmetric to the set defined in 1°); 3°) the set of points z with $|x|\leqq 2a$, $y>-a$. In Π_a, $\Phi(z)$ is holomorphic and satisfies, by lemma 2, the inequalities

$$(58)\qquad \left|\Phi(z)-\sum_{k=1}^{m}\frac{d_k}{z^{\lambda_k+1}}\right|\leqq\frac{5AM_{q+1}}{a|z|^{q+1}},\quad(\lambda_m\leqq q<\lambda_{m+1},\ m\geqq 1),$$

$$|\Phi(z)|\leqq\frac{5A}{a}.$$

Let P_a^s be the domain in the s-plane ($s=\sigma+it$), obtained by the transformation $z=-ie^s$, from Π_a with $t=0$ corresponding to $x=0$, $y<0$. If $\mathcal{C}E$ is composed of infinitely many intervals I_j corresponding to $x>0$ (and $-I_j$ corresponding to $x<0$) with left extremities a_j, $\log(a_j+a)-\log a_j\sim aa_j^{-1}$, $(j\to\infty)$. This, together with the fact that $V(x)$ is of bounded variation, shows that there exists a function $V_a(\sigma)$ ($\sigma\geqq\log 2a$), of bounded variation, such that the domain Δ^a defined by $|t|<\pi V_a(\sigma)$ is a subdomain of P_a^s, and such that

$$(59)\qquad \int|V(e^u)-V_a(u)|\,du<\infty.$$

Let us now set $F(s)=\Phi(-ie^s)$. In Δ^a, $F(s)$ is holomorphic, and satisfies there the inequalities

$$\left| F(s) - \sum_{k=1}^{m} i^{\lambda_n+1} d_n e^{-(\lambda_n+1)s} \right|$$

$$\leqq 5AM_{q+1}a^{-1}, \qquad (\lambda_m \leqq q < \lambda_{m+1},\ m \geqq 1),$$

$$|F(s)| \leqq 5Aa^{-1}.$$

On the other hand, it follows from (59) that, if in (57), $V(e^u)$ is replaced by $V_a(u)$ the integral still diverges. We may now apply theorem 1 (the adherence theorem), in exactly the same way we did to prove theorem 3, in order to get the desired result. In other words, we apply theorem 1, for each given a, on choosing the circle $C(s_0, \pi R) = C(\sigma_0, \pi R)$ as far to the left as compatible with the choice of a. We obtain, by theorem 1, an inequality for $|d_n|$; this inequality being such that, when a tends to zero, it shows that $d_n = 0$, $(n \geqq 1)$. A simple use of the solution of Watson's problem, mentioned many times, proves then that $F(s) \equiv 0$, that is to say, $\Phi(z) \equiv 0$, which proves the theorem. Of course in the statement of theorem 16, $D^*(\lambda)$ and D^* may be replaced by $\overline{D}^*(\lambda)$, $\overline{D}^*$.

If $E = R$, the statement of theorem 17 becomes very simple, since then $V(x) \equiv 1/2$. But, in the general case, we would be interested to take $V(x)$ for $x \in \mathcal{C}E$ as large as possible, since a greater magnitude of $V(x)$ increases the integrand in (57). But the fact that we require, in the statement, V to be of bounded variation, and equal to $1/2$ for $x \in E$, limits very much its magnitude. Thus it seems important to give weaker conditions on the regularity of V. We mention that the condition of bounded variation can be replaced, in theorem 1, and hence in theorem 17, for instance, by the following one: V has a continuous derivative V', such that $xV'(x)$ is of bounded variation, and such that $x[V'(x)]^2 \in L$ on $[0, \infty)$. Lemma M_4 in [2] can be applied to prove such an improvement of theorem 1, and consequently of theorem 17. See [10].

In [2] we have established conditions in order that the function $M(r) = \inf_n M_n/r^n$ be associated with a set E, or, what amounts to the same thing, in order that $\mathfrak{M}$ be associated with E (i.e., with E and $\Lambda =$ void).

When the set E is specialized, much better conditions can be

established. Let us mention the following interesting theorem, which is a corollary of a theorem of J. P. Kahane [11]. If $A \subset R$, we shall denote by $\mu(A; \sigma)$ the measure of $A \cap [0, \sigma]$.

THEOREM 18. *If E is a closed symmetric set on R, if, on denoting by $\mathcal{E}$ the set of the quantities σ such that $e^{\sigma} \in E$, the set $\mathcal{C}\mathcal{E}$ is a union of segments of length T, and if, on setting*

$$C(\sigma) = \sup_n (n\sigma - \log M_n), \tag{60}$$

we have

$$\int^{\infty} C(\sigma) \exp\left[-\mu(\mathcal{E}; \sigma) - 2\left(\frac{2}{\pi}\right)^{2/3} T^{-1/3}\mu(\mathcal{C}\mathcal{E}; \sigma)\right] d\sigma = \infty,$$

then $\mathfrak{M}$ is associated with E.

In particular:

THEOREM 19. *If $\mathcal{E}$ is a union of segments of infinitely increasing length (E being symmetric), and if, for an $\epsilon > 0$:*

$$\int^{\infty} C(\sigma) \exp\left[-\mu(\mathcal{E}; \sigma) - \epsilon\mu(\mathcal{C}\mathcal{E}; \sigma)\right] d\sigma = \infty,$$

then $\mathfrak{M}$ is associated with E.

For instance, if $E = \bigcup_{n \geq 1} \pm [K^{2n}, K^{2n+1}]$, $(K > 1)$, the factor of $C(\sigma)$ in the integrand, in the statement of theorem 18, is $e^{-(1/2+\epsilon_K)\sigma}$, with $\epsilon_K = O[(\log K)^{-1/3}]$. It follows from a result of Katznelson [12] that here ϵ_K can be taken of the order of magnitude $O(K^{-1} \log K)$.

8. We shall now give some new composition theorems established by Malliavin [13]. For the proof of these theorems we send the reader to the original paper, the methods used there being different from those we have used until now in this Pamphlet. We shall denote by $W(\mathfrak{M}, K(r))$, $K(r)$ being a positive function, the class of functions $F(z)$, holomorphic for $x > 0$, and satisfying there the inequalities: $|F(z)| \leqq M_n e^{-zK(r)}$, $n = [x]$, $r = |z|$.

THEOREM 20. *Let $\mathfrak{M}$, $\mathfrak{M}'$ be two sequences of positive numbers, and let $K_1(r)$ and $K_2(r)$ be two positive functions. If, on setting*

$$K^*(r) = \inf_{u \geqq r} [K_1(u) + K_2(u)]; \quad C^*(\sigma) = \sup_n (n\sigma - \log M_n M_n'),$$

the relationship

$$\int^{\infty} C^*(K^*(r) - a) \frac{dr}{r^2} = \infty,$$

holds for every $a > 0$, *then one of the classes* $W(\mathfrak{M}, K_1(r))$, $W(\mathfrak{M}', K_2(r))$ *contains only the function identically zero.*

In particular, the conclusion holds, if

$$\liminf_{r=\infty} [K_1(r) + K_2(r) - \log r] > -\infty,$$

and

$$\int^{\infty} C^*(\sigma) e^{-\sigma} d\sigma = \infty.$$

The last condition is equivalent to

$$\sum \frac{M_n M_n'}{M_{n+1} M_{n+1}'} = \infty$$

if $\log M_n$ and $\log M_n'$ are convex functions of n [1]. This theorem is an immediate consequence of the following theorem [13].

THEOREM 21. *Let* $K(r)$ *be a positive function, and let us set* $K.(r) = \inf_{r \geqq u} K(u)$. *If, with* $C(\sigma)$ *defined as in* (60),

$$\int^{\infty} C[K.(r) - a] \frac{dr}{r^2} = \infty$$

for every $a > 0$, *then,* $W(\mathfrak{M}, K(r))$ *contains only the function identically zero.*

The proof of this theorem will not be given here. In spite of the fact that theorem 20 is only a transcription of theorem 21 for a product of two functions, its interest lies in the following "equivalence theorem," [13]. Λ being a sequence of positive numbers, we shall set:

$$\lambda(r) = 2 \sum_{\lambda_n < r} \frac{1}{\lambda_n}, \qquad (r > 0).$$

THEOREM 22. *Let $\mathfrak{M}$ be a sequence of positive numbers, and Λ a sequence of positive integers. The two statements* 1°) *$L^\infty(\mathfrak{M}, \Lambda)$ contains no other odd infinitely differentiable function than the function identically zero*; 2°) *$W(\mathfrak{M}, \Lambda)$ contains no other function than the function identically zero, are equivalent.*

Theorem 20 may then be translated in the following way:

THEOREM 23. *Let Λ and Λ' be two sequences of positive odd integers, and let*

$$\lambda(r) = 2\sum_{\lambda_n<r}\frac{1}{\lambda_n}, \qquad \lambda'(r) = 2\sum_{\lambda_n'<r}\frac{1}{\lambda_n'}, \qquad (\lambda_n \in \Lambda, \lambda_n' \in \Lambda').$$

Let $\mathfrak{M}$, $\mathfrak{M}'$ be two sequences of positive numbers. If, on setting

$$C^*(\sigma) = \operatorname*{Sup}_{n\geqq 1}(n\sigma - \log M_n M_n'), \qquad \lambda^*(r) = \lambda(u) + \lambda'(u),$$

the relationship

$$\int^\infty C^*(\lambda^*(r) - a)\frac{dr}{r^2} = \infty \tag{61}$$

holds for every $a>0$, then one of the two classes $L^\infty(\mathfrak{M}, \Lambda)$, $L^\infty(\mathfrak{M}', \Lambda')$ contains no other odd functions than the function identically zero.

On using the same device as in the proof of theorem 10, we may state the following theorem:

THEOREM 24. *Let $\mathfrak{M}$ be a set of positive numbers, and Λ and Λ' be two sequences of odd positive integers; let a_1 and a_2 be two real numbers, and suppose that the infinitely differentiable function f satisfies the conditions $f^{(\lambda_n)}(a_1)=0$ $(\lambda_n\in\Lambda)$, $f^{(\lambda_n')}(a_2)=0$, $(\lambda_n' \in\Lambda')$, $f(a_1)=f(a_2)=0$. If the condition* (61) *is satisfied, then f is a periodic function, the period being one of the numbers $4a_1$, $4a_2$.*

As Malliavin has pointed out (orally), in this theorem one does not have to suppose that the sequences Λ, Λ' are without common term. One sees easily how to generalize theorem 24 to N constants $a_1, a_2, \cdots, a_N$ and N sequences of odd positive numbers $\Lambda_1, \Lambda_2, \cdots, \Lambda_N$.

Chapter III

EQUIVALENT PROBLEMS

1. Our next purpose is to show the equivalence between some problems related to the problem of "associating a sequence with a set," as defined above. Some of these problems were already treated in my previous Rice Institute Pamphlet [2], but we shall give here results which may be considered as reciprocal to those established in that monograph.

We introduce the following notations. $\mathfrak{M}$ being a sequence of positive numbers, $E \subset R$ a closed set, and $a > 0$, we denote by $\mathcal{R}_\delta\{\mathfrak{M}; E; a\}$ a relationship between $\mathfrak{M}$, E and a, which implies that for every infinitely differentiable function f on R, with $f \in L(\mathfrak{M})$, $[L(\mathfrak{M}) = L^1(\mathfrak{M})]$, $\sigma(f) \subset E_a = \bigcup_{x \in E} [x-a, x+a]$, the relationship holds: $\tau(f) = \delta(f)$,[13] where $\tau(f)$ is the set spanned (in the topology of L) by finite linear combinations of the form $a_1 f(x+\xi_1) + a_2 f(x+\xi_2) + \cdots + a_N f(x+\xi_N)$, and where $\delta(f)$ is the set spanned by finite linear combinations of the form $b_0 f(x) + b_1 f'(x) + \cdots b_K f^{(K)}(x)$.

We denote by $\mathcal{R}_{\delta\tau}\{\mathfrak{M}; E; a\}$ a relationship between $\mathfrak{M}$, E and a, which implies that, if f is infinitely differentiable, with $f \in L(\mathfrak{M})$, if $g \in L$, and if on denoting by $\Omega(g)$ the set on which the Fourier transform of g is zero, we have $\sigma(f) \cap \Omega(g) \subset E_a$, then $\tau(f) \subset \delta\tau(f, g)$, $\delta\tau(f, g)$ being the set spanned, in L, by the linear combinations of the form $a_0 f(x) + a_1 f'(x) + \cdots + a_N f^{(N)}(x) + C_1 g(x+\xi_1) + \cdots + C_P g(x+\xi_P)$. The results in [2] show that the properties $\mathcal{R}_\delta$, $\mathcal{R}_{\delta\tau}$ furnish interesting results on the approximation of continuous functions on a set E, by weighted polynomials.

We denote by $\mathcal{R}_H\{\mathfrak{M}; E; a\}$ a relationship between $\mathfrak{M}$, E and a, which implies that $\mathfrak{M}$ is associated (∞) with E_a; that is to say, that each $\Phi(z)$, holomorphic and uniform outside of E_a, and satisfying there the inequalities $|\Phi(z) z^n y| \leqq M_n (n \geqq 0)$ is identically zero.

It is clear that a property $\mathcal{R}_{\delta\tau}$ is a property $\mathcal{R}_\tau$. But it is interesting, and important for applications, to notice that the reciprocal is (with some restrictions) also true. We are, indeed, in a position to prove the following theorem [14].

THEOREM 25. *If E contains the origin, each property $\mathfrak{R}_i\{\mathfrak{M}; E; a\}$ is a property $\mathfrak{R}_{ir}\{\mathfrak{M}; E, a'\}$ for every a' with $0<a'<a$.*

This theorem is equivalent, by the Banach-Riesz theorem, to the following theorem:

THEOREM 26. *If $0\in E$, $f\in L(\mathfrak{M})$, if $h\in L^\infty$, if $g\in L$, if the functions $f^{(n)}(-x)(n\geqq 0)$ are orthogonal to $h(x)$, if $g*h=0$, if $[\sigma(f)\cap\Omega(g)]\subset E_{a'}$ and if $\mathfrak{R}_i\{\mathfrak{M}; E; a\}$ is satisfied, then also $f*h=0$.*[14]

We now prove this theorem.

Let the function $p(x)$ be defined in the following fashion:

$$p''(x) = f*h = \int f(x-u)h(u)du, \qquad p(0) = p'(0) = 0. \tag{62}$$

We have for $n\geqq 2$:

$$p^{(n)}(x) = f^{(n-2)} * h. \tag{63}$$

From the orthogonality of $f^{(n)}(-x)$ to $h(x)$, for every $n\geqq 0$, and from $p(0)=p'(0)=0$ it follows that

$$p^{(n)}(0) = 0, \qquad (n \geqq 0). \tag{64}$$

It follows also form (63) that

$$|p^{(n)}(x)| \leqq M_{n-2}\|h\| = AM_{n-2}, \qquad (n \geqq 2), \tag{65}$$

where A is a constant.

If $\liminf_{n=\infty} M_n^{1/n} < \infty$, then $p''(x)\equiv 0$ (see [1] for the theorem of quasi-analyticity), and our theorem is proved. If $\lim M_n^{1/n} = \infty$, we may then replace M_n, in (65), by:

$$|p^{(n)}(x)| \leqq AB^n M_{n-2}^c, \qquad (n \geqq 2), \tag{66}$$

where $\{\log M^c{}_n\}$ is the convex regularized sequence of $\{\log M_n\}$ (see the theorem of Gorny in [1]), and where B is a constant.

Let us set $\alpha(x)=(\sin^6 x)x^{-6}$, and, c being a positive constant,

$$\begin{aligned} \phi_c(x) &= p(x)\alpha(cx), \\ \psi_c(x) &= \phi_c(x) - \phi_c''(x). \end{aligned} \tag{67}$$

Let $A_c(u)$ be the Fourier transform of $\alpha(cx)$. One sees immediately that the support of $A_c(u)$ is the interval $[-6c, 6c]$,

and that $A_c(u)$ has 4 continuous derivatives; therefore:

$$(-i)^p c^n \alpha^{(n)}(cx)x^p = \frac{1}{\sqrt{2\pi}} \int_{-6c}^{6c} [A_c(u)u^n]^{(p)} e^{iux} du, \quad (0 \leqq p \leqq 4),$$

which shows that, for c sufficiently small $(6c<1)$:

$$c^n \left| \alpha^{(n)}(cx) \right| \left| x \right|^p \leqq M_c(6c)^n, \qquad (0 \leqq p \leqq 4),$$

where M_c is a constant.[15] Since, from (65), it follows also that $p(x)=0(x^2)$, $p'(x)=0(x)$, $(|x| \to \infty)$, and, since there exists a constant $K>0$ such that $M^c_{n-2} \leqq K M^c_n$, $(n \geqq 2)$, we see that, if c is sufficiently small, then on setting $\xi(x) = M\psi_c(x)$, where M is a suitably chosen constant, we may write

$$\xi^{(n)} \in L, \ (n \geqq 0), \qquad \left\| \xi^{(n)} \right\|_1 \leqq M_n^c \leqq M_n. \tag{68}$$

We have also:

$$\xi(x) * e^{-|x|} = M\phi_c(x), \tag{69}$$

since the Fourier transform of $\xi(x)*e^{-|x|}$ is $(1+u^2)^{-1}R(u)$, where $R(u)$ is the Fourier transform of ξ, and $R(u)=(1+u^2)Q(u)$, where $Q(u)$ is the Fourier transform of $M\phi_c(x)$; in other words, we may write, by (69):

$$\phi_c^{(n)}(0) = \int \xi^{(n)}(-y)e^{-|y|} dy = 0, \qquad (n \geqq 0). \tag{70}$$

We shall now prove that, if c is sufficiently small, then $\sigma(\xi) \subset E_a$. From $p''=f*h$ and $g*h=0$ it follows that $g*p''=0$. From the theorems proved in [2] it follows then that $\sigma(p'') \subset \sigma(f) \cap \Omega(g)$.

Let (P^+, P^-) be the Fourier-Carleman transform of p, that is to say:

$$P^+(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} p(u)e^{-iuz} du, \qquad y > 0,$$

$$P^-(z) = \frac{-1}{\sqrt{2\pi}} \int_{0}^{\infty} p(u)e^{-iuz} du, \qquad y < 0.$$

These two functions are holomorphic in the corresponding half-planes, since $p(x)=O(x^2)$.

Since $p(0)=p'(0)=0$, we get by two integrations by parts of both integrals:

$$\sqrt{2\pi}P^+(z) = -\frac{1}{z^2}\int_{-\infty}^{0} p''(u)e^{-iuz}du, \qquad y > 0,$$

$$\sqrt{2\pi}P^-(z) = \frac{1}{z^2}\int_{0}^{\infty} p''(u)e^{-iuz}du, \qquad y < 0.$$

Hence, P^+ can be continued analytically into P^- across any interval not containing points of $\sigma(p'')$, except, perhaps, across an interval containing the origin. We have for $y>0$:

$$\sqrt{2\pi}[P^+(x+iy) - P^-(x-iy)] = -\int p(u)e^{-y|u|}e^{-iuz}du,$$

and

$$\int_{t-6c}^{t+6c} [P^+(x+iy) - P^-(x-iy)]A_c(t-x)dx = \int p(u)e^{-y|u|}\alpha(cu)e^{-iuz}du.$$

If $a'<a'+6c<a''<a$, and if $t\in \mathcal{C}E_a$, then the integrand tends to zero with y, uniformly with respect to y. Therefore, in such an interval

$$\int p(u)\alpha(cu)e^{-iuz}du = 0;$$

and since $0\in E$, we see that, $\sigma(\phi_c)\subset E_{a''}$ for $c>0$ sufficiently small. If we consider the Fourier-Carleman transform of ϕ_c and integrate twice by parts the expressions furnishing the functions of this transform, we see (again on taking in account the fact that $\phi_c(0)=\phi_c'(0)=0$, and that $0\in E$) that $\sigma(\phi''_c)\subset E_a$. Hence, $\sigma(\xi)\subset E_a$, if c is sufficiently small.

Since we suppose that $\mathfrak{R}_i\{\mathfrak{M}; E, a\}$ is satisfied we see from the properties of ξ [(68), (70) and $\sigma(\xi)\subset E_a$] that $\xi(x)*e^{-|x|}=0$, that is to say, that $\phi_c\equiv 0$; therefore also $p\equiv 0$, hence $f*h=0$, and the theorem is proved.

2. If $f\in L$, $\phi\in L$, then also $f*\phi\in L$; since, on the other hand,

every function Φ continuous on R with $\lim_{|u|=\infty}\Phi(u)=0$ is a uniform limit of the Fourier transforms of functions belonging to L, we see immediately that, if F is the Fourier transform of a function $f\in L\{\mathfrak{M}\}$ with $F(u)=0$, for $u\in\mathfrak{C}E_a$, the property $\mathfrak{R}_\delta\{\mathfrak{M};\ E;\ a\}$ implies that to every $\epsilon>0$ corresponds a polynomial $P(u)$ such that $|\Phi(u)-P(u)|\,|F(u)|<\epsilon$, (see, for instance [2]). But theorem 25 shows that the property $\mathfrak{R}_\delta\{\mathfrak{M};E;a'\}$ with $a'<a$ implies already such a weighted approximation, without being obliged to suppose $F(u)=0$ on E_a. Indeed, such an approximation is possible, as it is again proved in [2] if the property $\mathfrak{R}_{\delta\tau}\{\mathfrak{M};\ E;\ a\}$ is satisfied. This remark shows even more the interesting aspects of the following theorem:

THEOREM 26. *If E contains the origin, each property $\mathfrak{R}_\delta\{\mathfrak{M};E;a\}$ is a property $\mathfrak{R}_H\{\mathfrak{M};\ E;\ a'\}$ for each a' with $0<a'<a$.*

The converse of this theorem was already proved in [2] (with of course, a and a' interchanged).

Let Φ be a holomorphic and uniform function in $\Delta_{E_{a'}}$ (i.e. outside of $E_{a'}$), and let us set

$$\phi_1(u)=\int_{-\infty+iy}^{\infty+iy}\Phi(z)e^{iuz}dz, \qquad (y>0). \tag{71}$$

If the property $\mathfrak{R}_\delta\{\mathfrak{M};\ E;\ a\}$ is satisfied, this integral has a meaning, and by Cauchy's theorem, we see that it is independent of $y(y>0)$. $\phi_1(u)$ is infinitely differentiable and we have, for $n\geqq 0$:

$$\phi_1^{(n)}(u)=i^n\int_{-\infty+iy}^{\infty+iy}\Phi(z)\,z^n e^{iuz}dz,$$

with

$$|\phi_1^{(n)}(u)|\leqq e^{-uy}\int_{-\infty+iy}^{\infty+iy}|\Phi(z)|\,|z|^n dx$$

$$\leqq e^{-uy}M_{n+2}y^{-1}\int\frac{dx}{|x+iy|^2}.$$

For $u>0$, $\phi_1(u)=0$. Thus:

$$\phi_1^{(n)}(0)=0, \qquad (n\geqq 0).$$

For $u \leqq 0$, we get, on setting $y = -u^{-1}$:

$$(72) \quad |\phi_1^{(n)}(u)| \leqq eM_{n+2}|u| \int \frac{dx}{|x - iu^{-1}|^2} = AM_{n+2}|u|^2, \qquad (n \geqq 0),$$

where A is a constant. (These first lines of the proof are similar to those of Ostrowski's proof for the equivalence of Watson's problem and the problem of quasi-analyticity [3]).

The same integral (71), taken for $y<0$, defines a function $\phi_2(u)$, infinitely differentiable on R with $\phi_2^{(n)}(0) = 0 (n \geqq 0)$, $\phi_2(u) = 0$ for $u \leqq 0$, and satisfying for $u \geqq 0$ the inequalities (72). Thus, the function ϕ, equal to ϕ_1 for $u \leqq 0$, and equal to ϕ_2 for $u \geqq 0$, has the following properties: ϕ is infinitely differentiable on R, $\phi^{(n)}(0) = 0$ $(n \geqq 0)$, and, on R:

$$|\phi^{(n)}(u)| \leqq AM_{n+2}|u|^2, \qquad (n \geqq 0).$$

By methods very similar to those used in the proof of theorem 26, we see that, for $c>0$ sufficiently small, the function

$$g_c(x) = \alpha(cx) \int_0^x dx_1 \int_0^{x_1} dx_2 \int_0^{x_2} dx_3 \int_0^{x_3} \phi(t)dt$$

satisfies the following conditions:

$$g_c^{(n)}(0) = 0 (n \geqq 0), \qquad g_c^{(n)} \in L (n \geqq 0);$$

$$\|g_c^{(n)}\|_1 \leqq kM_{n-2} (n \geqq 2), \qquad \sigma(g_c) \subset E_{a''},$$

where $a' < a'' < a$, k constant. Let us set $\gamma_c(x) = M[g_c(x) - g_c''(x)]$ Again, as in the proof of theorem 26, we see that $Mg_c = \gamma_c * e^{-|x|}$, that $\gamma_c^{(n)}(-x)$ is orthogonal to $e^{-|x|}$, and that, if M is chosen sufficiently small, then $\|\gamma^{(n)}\|_1 \leqq M_n$, $(n \geqq 0)$. Since $\mathcal{R}_\delta\{\mathfrak{M}: E; a\}$ is satisfied, we get $g_c = \gamma_c * e^{-|x|} \equiv 0$. Which proves that $\phi \equiv 0$, that is to say, $\Phi \equiv 0$; and this proves the theorem.

3. The sequence $\mathfrak{M}$ and the set $E \subset R$ given, it is not necessary to consider a-coverings of E, if we wish to establish a one sided relationship between the class of functions Φ, holomorphic outside of E, and such that $|\Phi(z)z^n y| \leqq M_n (n \geqq 1)$, and the class of infinitely differentiable functions f with $f \in L(\mathfrak{M})$, $\sigma(f) \subset E$.

In [2] the following facts are proved:

1°) $\delta(f) \subset \tau(f)$ for every $f \in L^p(\mathfrak{M})$. The proof of this statement is immediate.

2°) If $\mathfrak{M}$ is associated (∞) with E, that is to say, if the only function holomorphic outside of E, and satisfying the inequalities $|\Phi(z)z^n y| \leqq M_n (n \geqq 1)$, is the function identically zero, then $\tau(f) \subset \delta(f)$ [hence $\delta(f) \equiv \tau(f)$], for each function f with $f \in L(\mathfrak{M})$, $\sigma(f) \subset E$.

But, we were obliged to consider a-coverings in order to establish a "converse" of the fact 2°), that is to say theorem 25. As we have seen in [2], the statement 2°) leads to the following one:

3°) If $\mathfrak{M}$ is associated (∞) with E, and if F is the Fourier transform of a function f, with $f \in L(\mathfrak{M})$, $\sigma(f) \subset E$, then to each function Ψ, continuous on R with $\lim \Psi(u) = 0$ $(|u| \to \infty)$, and each $\epsilon > 0$, corresponds a polynomial P such that $|\Psi(u) - P(u)| |F(u)| < \epsilon$ (on page 59 we have mentioned such a statement, where a covering of E is involved).

But it is interesting to notice that 3°) also admits a converse, provided a-coverings of E are introduced. Such theorems are established in [15]. The full treatment of the problem of equivalence between the problems "$\mathfrak{M}$ associated with E," "$\tau(f) = \delta(f)$," "$\{F(x)x^n\}$ complete on R'' will be given in another monograph.

NOTES

1. A channel of diameter $2K$ is the union of circles $C(s, K)$, the centers s of these circles running over a simple Jordan arc J. We say that F can be continued analytically from Δ to $C(s_0, \pi R)$ through a channel of diameter greater than $2\pi\overline{D}^{\cdot}$ if there exists a channel of diameter $2K > 2\pi\overline{D}^{\cdot}$, the end-points of J being s_0, s_1 with $C(s_1, \pi R) \subset \Delta$, and if there exists a function holomorphic in the union of $C(s_0, \pi R)$, the channel and Δ.
2. p may be equal to $+\infty$.
3. The symbol $\int$, without indiction of bounds, means that the integral is extended over $R(\int = \int_{-\infty}^{\infty})$.
4. Q being a set, $\overline{Q}$ demotes its closure.
5. The definition of $L^2(\mathfrak{M}, \Lambda)$ (as that of $L^p(\mathfrak{M}, \Lambda)$, in general) implies, among other conditions, that, if $\phi \in L^2(\mathfrak{M}, \Lambda)$, then $\phi(0) = 0$. But, in the proof of theorem 7, we do not use this fact. In other words, here, for ϕ we have to suppose only that $\phi^{(\lambda_n)}(0) = 0$, $(\lambda_n \in \Lambda)$; and, of course, that $\phi^{(n)} \in L^2(n \geq 0)$, $\|\phi^{(n)}\|_2 \leqq M_n (n \geq 1, \{M_n\} = \mathfrak{M})$.
6. This case of Watson's problem is near to the classical case, due to the convergence of the integral in (21), and it was treated already in [6].
7. The device we are using here was already used in [7].
8. It is possible to establish a certain duality principle by using the Mellin transform in order to make correspond to each function $\Phi(z)$ having an asymptotic representation in the half-plane $y > 0$, a function ϕ, infinitely differentiable on $[0, \infty)$, and having bounded derivatives. The orders of the derivatives vanishing at the origin are the exponents of the asymptotic series.
9. A and B being two sets of numbers, $A \oplus B$ denotes the set of the numbers of the form $\alpha + \beta$, where α is any number belonging to A, and β any number belonging to B.
10. We may take for α, for instance, the function

$$\alpha(z) = \frac{1}{z}\left[(z+1)^{1/2} - 1\right]^2,$$

with the real determination of $(z+1)^{1/2}$ for $z = x > -1$.
11. If the set $\mathfrak{N}$ contains only μ_k, the product of this formula should represent the constant 1.
12. We recall that $\omega(0) = 0$.
13. It is established in [2] that, for every $p \geq 1$, $\delta(f) \subset \tau(f)$. This fact is very elementary.
14. Since then it will follow, by the Banach-Riesz theorem, that $\tau(f) \subset \delta(f)$; this together with $\delta(f) \subset \tau(f)$, (see note 13) will then prove theorem 25.
15. M_q is independent of n. It does depend on c.

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